Gaps, Gluts and Fuzziness

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In this paper I firstly recollect some of the motivations for many-valued logics, both those such as the Łukasiewicz fuzzy logic (Ł$_{\aleph}$), which accommodates intermediate degrees of truth and those which treat extra truth values as truth value gaps or gluts, such as the Strong Kleene Logic (K_3) and the Logic of Paradox (LP). I then develop a type of logical system which is essentially a combination of Ł$_{\aleph}$ and *FDE*, consider conditionals in such a type of logic and sketch some possible applications. Prior to concluding, I sketch how this *fuzzy FDE* can be used as the basis of a basic fuzzy relevant logic.

Many-Valued Logic

Many-valued logics extend classical logic by adding extra truth values, beyond the classical two: *true* and *false*. There are various motivations for doing so, following are some main ones.

Fuzzy Logic

Fuzzy logic and fuzzy set theory reject the classical dichotomy between truth and falsity and allow for degrees of truth and set membership.

There are two senses in which the term fuzzy logic is used. Fuzzy logic in the broad sense denotes the paradigm of fuzzy reasoning which has extensive application in artificial intelligence, natural language analysis and engineering control systems which control physical world events and need to be sensitive to changes of degree in environmental factors. One of the philosophically interesting applications of fuzzy logic is to the issue of vagueness. Fuzzy logic in the narrow sense is a branch of many-valued symbolic logic with a fuzzy notion of truth, developed in the spirit of classical logic. It is concerned with the development of formal systems with fuzzy truth values and the study of the types of properties of formal systems which would interest a logician.[1]

One of the most important formalisations of fuzzy logic in this narrow sense is the fuzzy logic Ł$_{\aleph}$, which generalises the Łukasiewicz many-valued logics. The connectives of this family of logics are defined as follows:

$$f_{\neg}(x) = 1 - x$$

$$f_{\wedge}(x, y) = min(x, y)$$

$$f_{\vee}(x, y) = max(x, y)$$

$$f_{\rightarrow}(x, y) = min(1, 1 - x + y)$$

[1] Petr Hájek, 'Fuzzy Logic', *The Stanford Encyclopedia of Philosophy* (Fall 2002 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2002/entries/logic-fuzzy/#1 >.

The set of truth values for $Ł_{\aleph}$ are the real numbers between 0 and 1, written as [0,1]. 1 is the only designated truth degree. A more general fuzzy logic, Ł, can be obtained by the addition of a variable δ to the language, such that $\delta \in [0,1]$ and the set of designated values, $\mathcal{D}_\delta$, is $\{x | x \geq \delta\}$.[2]

Truth Value Gaps

Another motivation for considering truth values beyond those of classical logic are truth-value gaps. One place where the simple dichotomy of truth and falsity seems to come unstuck is in situations involving future contingents. For example, it seems that a statement such as 'The 35th prime minister of Australia will be a woman' is neither true nor false.

Another reason for supposing there are gaps are denotation failures. For example, some have argued that statements containing a non-denoting name, such as 'The present king of France is bald', have no truth value. Similarly, statements containing category mistakes, such as '10 is a loud number' or 'The capital of Australia is 10', might be truth valueless or meaningless, so that they must be assigned a truth value alternative to the classical ones of *true* and *false*

The logic K_3, philosophically motivated by consideration of truth-value gaps, introduces a third truth value n to classical logic, to be thought of as *neither true nor false.* The truth tables for its connectives, $\neg$, $\wedge$ and $\vee$ and $\rightarrow$ are defined as follows:

$f_\neg$	0
1	0
n	n
0	1

$f_\wedge$	1	n	0
1	1	n	0
n	n	n	0
0	0	0	0

$f_\vee$	1	n	0
1	1	1	1
n	1	n	n
0	1	n	0

$f_\rightarrow$	1	n	0
1	1	n	0
n	1	n	n
0	1	1	1

The sole designated value for this logic is 1. The logic, $Ł_3$, replaces the K_3 implication connective with one given by the following truth table:

$f_\rightarrow$	1	n	0
1	1	n	0
n	1	1	n
0	1	1	1

Truth Value Gluts

Another motivation for considering truth values beyond those of classical logic are truth-value gluts. One reason for supposing that there are truth-value gluts concerns the paradoxes of self-reference. A classic example is the barber paradox: A village has a barber in it, who shaves all and only the people who do not shave themselves. Who shaves the barber? If he shaves himself, then he does no, but if he does not shave himself, then he does. This argument seems sound yet has a conclusion of the form $B \wedge \neg B$, so a suitable logic might be one which tolerates such contradictions.

Another reason concerns inconsistent situations. In many cases it ought to be possible to reason with inconsistent information in a controlled and discriminating way. For example, a computer game might have a set of rules for the behaviour of a type of character in the game. A modification of

[2]Graham Priest, *An Introduction to Non-Classical Logic* (Volume I), Cambridge, Cambridge University Press, 2001, p. 216.

the character's abilities in a subsequent version of the game might mean that a new rule is added to this set which makes it inconsistent. Rather than succumbing to the classical principle *anything follows from a contradiction*, it is preferable that the system tolerate this inconsistent information in some suitable way, so that the game continues without collapsing into triviality.

The logic *LP*, philosophically motivated by consideration of truth-value gluts, introduces a third truth value b to classical logic, to be thought of as *both true and false.* The truth tables for its connectives, $\neg$, $\wedge$ and $\vee$ and $\rightarrow$ are the same as those for K_3, replacing n with b. The two designated values for this logic are 1 and b.

The logic, RM_3, replaces the *LP* implication connective with one given by the following truth table:

$f_\rightarrow$	1	b	0
1	1	0	0
b	1	b	0
0	1	1	1

FDE

The logic *first degree entailment (FDE)* is essentially a combination of the 3-valued logics such as K_3 and L_3 which treat their third truth value as a gap and the 3-valued logics such as *LP* and RM_3 which treat their third truth value as a glut. A simple many-valued semantics can be given for this logic, resulting in a 4-valued logic with values 1, 0, b and n. The truth conditions for the connectives of this logic are given in the following truth tables:

$f_\neg$	
1	0
b	b
n	n
0	1

$f_\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$f_\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

Figure 1 depicts the diamond lattice corresponding to *FDE*.

An alternative semantics for this logic is a relational semantics. The basic idea is that instead of formulating interpretations as functions, mapping each proposition to one of the four truth values, they can be formulated as relations between propositions and the two classical truth values. Given the two truth values 1 and 0, there are four ways in which a formula may relate to them; it may relate to 1 and not 0 (1), 0 and not 1 (0), neither 1 nor 0 (n) and both 1 and 0 (b). This semantics better emphasises the fact that the two extra 'truth values' in *FDE* actually represent the absence or presence of both truth and falsity.

Gaps, Gluts and Fuzziness

Now, we have seen how many-valued logics can be used for logics in which the extra values represent intermediate degrees of truth (i.e. $Ł_{\aleph}$) or for logics in which the extra values represent truth-value

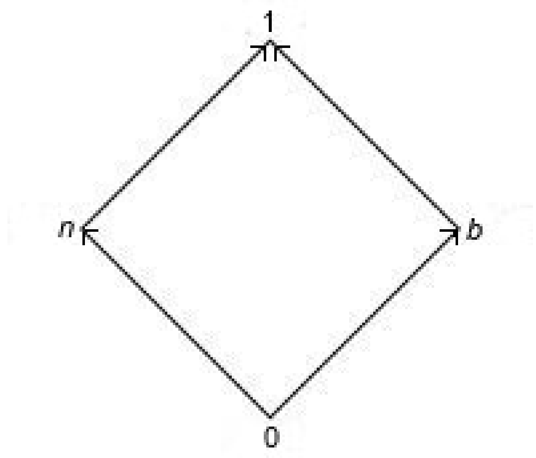

Figure 1: *FDE* diamond lattice

gaps and gluts (i.e. *FDE*). I now come to the development of a type of logic which essentially combines both of these rationales.[3]

Truth values in this type of logic can be represented with a tuple, (t, f), where $t, f \in [0, 1]$ and where from a relational perspective t represents the degree to which a formula relates to *true* and f represents the degree to which a formula relates to *false*. The lattice resulting from a conversion of the *FDE* truth values 1, 0, n and b to their tuple counterparts is depicted in figure 2.

This type of logic can be defined by the formal structure $\langle n, \mathcal{T}, \mathcal{V}, \mathcal{D}, \{\neg, \wedge, \vee\}\rangle$, where

- $n \in \mathbb{N}$ and $n \geq 2$
- $\mathcal{T}$ is a set consisting of the *truth value degrees*, such that $\mathcal{T} = \{i/(n-1) | 0 \leq i \leq n-1\}, |\mathcal{T}| = n$
- $\mathcal{V}$ is the set of truth values, such that $\mathcal{V} = \{(t, f) | t \in \mathcal{T}, f \in \mathcal{T}\}, |\mathcal{V}| = n^2$
- $\mathcal{D} \in \mathcal{V}$ and is the set of designated values

The semantic functions that correspond to the connectives $\wedge$, $\vee$ and $\neg$ are as follows:

$$f_{\neg}((t, f)) = (f, t)$$

$$f_{\vee}((t_1, f_1), (t_2, f_2)) = (max(t_1, t_2), min(f_1, f_2))$$

$$f_{\wedge}((t_1, f_1), (t_2, f_2)) = (min(t_1, t_2), max(f_1, f_2))$$

When $n = 2$ the resulting logic simply corresponds to *FDE*. When $n = 3$ the resulting system has 3 truth value degrees and 9 truth values. Its lattice is depicted in figure 3.

[3] I came across a sketch of this idea in [13]

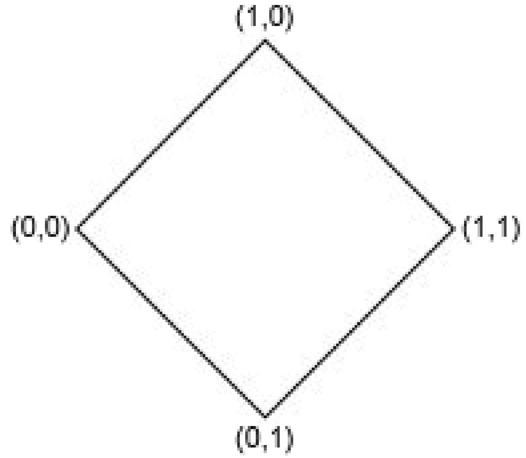

Figure 2: *FDE* lattice with truth value tuples

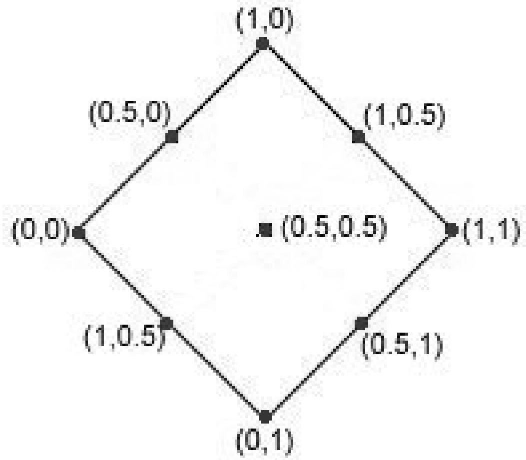

Figure 3: Lattice when $\mathcal{T} = 3$

Resulting is a logic of independent degrees of truth and falsity. Like $Ł_{\aleph}$ it has intermediate degrees of truth, yet unlike $Ł_{\aleph}$, in which truth and falsity are in diametrical opposition, the degree of truth can be adjusted without affecting the degree of falsity and vice versa. Extending this logic so that $\mathcal{T} = [0, 1]$ results in a *fuzzy FDE*. Remaining discussion will in general be based on this case.

Following are several depictions of regions that constitute the lattice for this logic.

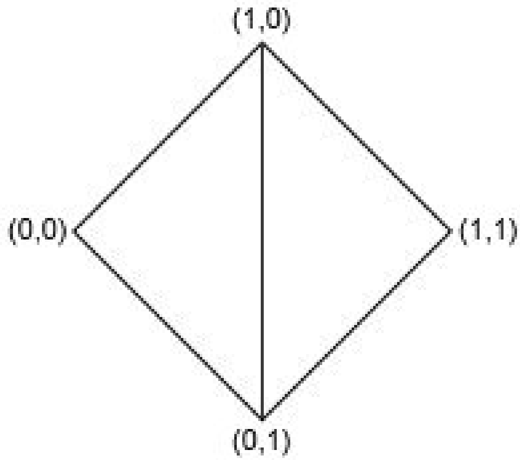

Figure 4: Well-Define Truth Values ($t + f = 1$)

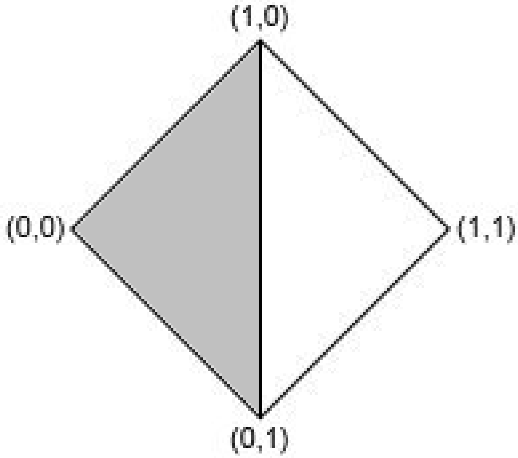

Figure 5: Under-Defined Truth Values ($t + f < 1$)

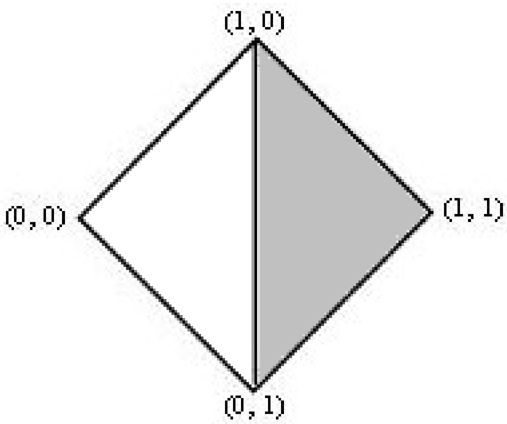

Figure 6: Over-Defined Truth Values ($t + f > 1$)

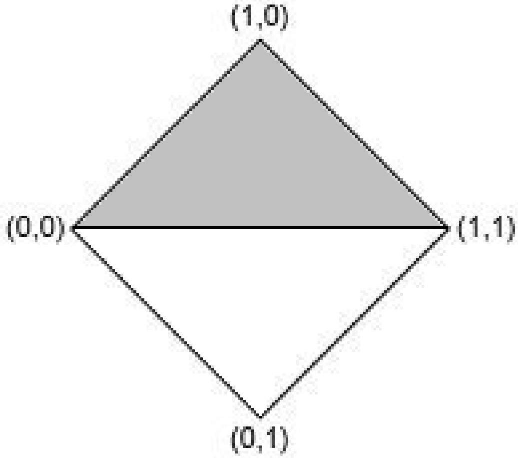

Figure 7: More True than False ($t > f$)

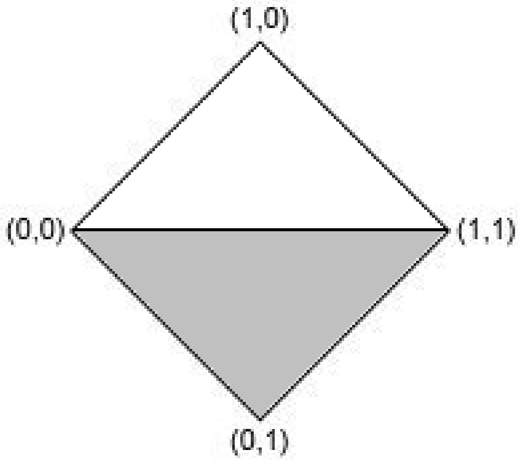

Figure 8: More False than True ($t < f$)

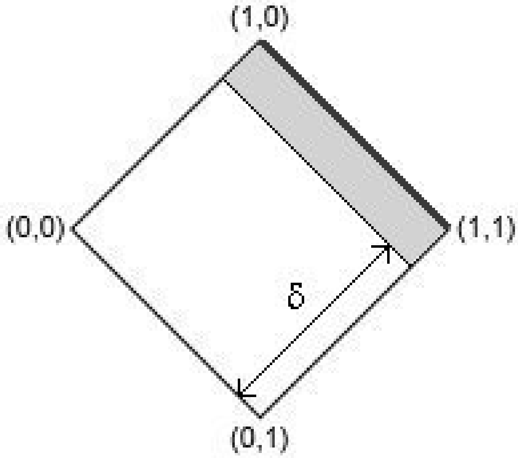

Figure 9: Designated Values

Semantic Consequence

An interpretation function v of this language, assigns to each proposition a truth value. v is composed of two sub-functions, v_t and v_f. The former assigns the t portion of the truth value tuple and the latter assigns the f portion of the truth value tuple.

In determining the designated values, (t, f), only the t portion is considered. Since fuzziness is an aspect of this type of logic, a number δ can be defined which determines the designated values. Taking the set of designated values, $\mathcal{D}_\delta$, to be $\{(t, f)|t \geq \delta\}$ (depicted in figure 9), we can proceed with a definition of logical consequence for a particular δ.

$\Sigma \vDash_\delta A$ iif for all interpretations, v, if $v_t(B) \geq \delta$ for all $B \in \Sigma$, then $v_t(A) \geq \delta$.

Abstracting from particular instances of $\vDash_\delta$, the general consequence relation is defined as follows:

$\Sigma \vDash A$ iff for all δ, where $0 \leq \delta \leq 1$, $\Sigma \vDash_\delta A$

An inference is valid if it preserves designated values. If Σ is a set of formulas, let $v_t[\Sigma]$ be $\{v_t(B)|B \in \Sigma\}$. The consequence relation is defined as:

$$\Sigma \vDash A \text{ iff for all } v,\ Glb(v_t[\Sigma]) \leq v_t(A)$$

Of course, as with the logic $Ł_\aleph$, a special case of L where $\mathcal{D} = \{1\}$, the simplest option, and that which I am interested in investigating further, is when $\mathcal{D} = \{(1, f)|0 \leq f \leq 1\}$. This region is indicated by the highlighted edge in figure 9. I shall refer to this type of logic as FDE_n for any given n and FDE_∞ when $\mathcal{T}$ = [0,1].

Implication

We now come to the task of thinking about implication in the context of FDE_∞. A straightforward though not so interesting possibility is the material implication, $\supset$, such that $A \supset B \equiv \neg A \vee B$. In the context of fuzzy logic, such implications are called *S-implications.*[4]. This material implication can be defined as follows:

$$f_\supset((t_1, f_1), (t_2, f_2)) = (max(f_1, t_2), min(t_1, f_2))$$

Another simple option would be to mirror the Łukasiewicz conditional:

$$f_\to((t_1, f_1), (t_2, f_2)) = (min(1, 1 - t_1 + t_2), 1 - min(1, 1 - t_1 + t_2))$$

More interesting conditionals can be found by taking into consideration more than just a comparison between the truth of the antecedent and truth of the consequent.

Conditional 1

One interesting notion of implication for a logic based on *FDE* is that of the logic *BN4*, which extends *FDE* by defining an implication operator that gives the following truth table:

$\to$	1	b	n	0
1	1	0	n	0
b	1	b	n	0
n	1	n	1	n
0	1	1	1	1

Given this, a particularly interesting possibility is to define a conditional for FDE_∞ such that

- when restricted to the well-defined values, (t, f), such that $t, f \in [0, 1]$ and $t + f = 1$, gives the Łukasiewicz implication operator.
- when restricted to the values $1(1, 0)$, $b(1, 1)$, $n(0, 0)$ and $f(0, 1)$ gives the *BN4* implication operator.[5]

A function suitable for defining such a conditional, $\to_1$, is as follows:

$$f_{\to_1}((t_1, f_1), (t_2, f_2)) = (min(t_1 \ominus t_2, f_2 \ominus f_1), t_1 * f_2)$$

where $x \ominus y$ is defined as:

$$min(1, 1 - x + y)$$

and $x * y$ is defined as:[6]

[4] Hájek, 2002.
[5] See http://consequently.org/news/2006/02/19/degrees_of_truth_degrees_of_falsity/index.php
[6] This binary operator is known as the Łukasiewicz t-norm. See [9]

$max(0, x + y - 1)$

The behaviour of this conditional can be seen as obeying the behaviour of the *BN4* conditional whilst at the same time being sensitive to the Łukasiewicz conditional and the degrees of truth and falsity involved. A formal demonstration of this is given in Appendix A.

Of implication in *BN4* Greg Restall writes

> The value "true" is in the set $a \to b$ just when if a is at least "true" then b is at least "true", and if b is at least "false" then so is a. On the other hand, a conditional $a \to b$ is at least "false" if a is at least "true" and b is at least "false".[7]

Relating this quote to the definition of $\to_1$ I have given, let us begin by looking at the definition for the t portion of the resulting truth value tuple. If either the truth of the antecedent is greater than the truth of the consequent or the falsity of the consequent is greater than the falsity of the antecedent, then the truth of the result is the difference between 1 and greater of these two falls. Otherwise the value of t is 1. For the f portion, its value is as good as the truth of the antecedent and the falsity of the consequent; the truer the antecedent and falser the consequent, the falser the resulting truth value.

Here is an example demonstrating the application of this conditional to values which neither lie on the four points of the diamond lattice nor the well-defined line, the result being a fuzzy over-defined value.

$$\begin{aligned} v((0.9, 0.7) \to_1 (0.8, 0.9)) &= (min(0.9 \ominus 0.8, 0.9 \ominus 0.7), 0.9 * 0.9) \\ &= (min(min(1, 1 - 0.9 + 0.8), min(1, 1 - 0.9 + 0.7)), max(0, 0.9 + 0.9 - 1)) \\ &= (0.8, 0.8) \end{aligned}$$

In going from antecedent to consequent, truth has been lost and falsity gained. Of this decrease and increase, the increase of falsity is greater, and since this increase is 0.2, truth should correspondingly be set to 0.8. Since the antecedent is 0.9 true and the consequent is 0.9 false, the degree of falsity is 0.8.

Let $BN4_n$ denote the logic obtained by adding $\to_1$ to FDE_n. It is clear that for any given n, if $\Sigma \models_{BN4_n} A$, then $\Sigma_{BN4} \models A$. For no given n (apart from 2) is it the case that the converse holds. For example, the formula $A \vee \neg B \vee (A \to B)$ is valid in *BN4* but not in $BN4_3$[8], hence not valid in $BN4_n$ for any $n > 3$

The fusion operator of *BN4* is defined as $\neg(A \to \neg B)$, giving the following truth table:

$\circ$	1	b	n	0
1	1	1	n	0
b	1	b	n	0
n	n	n	0	0
0	0	0	0	0

[7] Greg Restall, 'Relevant and Substructural Logics', in Dov Gabbay and John Wood (eds.). *Handbook of the History of Logic, Volume 7, Logic and the Modalities in the Twentieth Century*, Elsevier, 2006, p. 53.

[8] e.g. $v(A) = (0.5, 0), v(B) = (0.5, 0.5)$

Translating this to $\rightarrow_1$ gives the following definition:

$$f_\circ((t_1, f_1), (t_2, f_2)) = (t_1 * t_2, min(t_1 \ominus f_2, t_2 \ominus f_1))$$

The fusion operator of *BN4* is commutative and associative and is residuated by $\rightarrow$. I have verified that the first two of these properties hold for the fusion operator for $BN4_\infty$. I have not verified that it is residuated by $\rightarrow_1$.

Here are two other possible definitions for a conditional.

Conditional 2

Another possibility for defining a conditional is to basically take the definition for the truth of $\rightarrow_1$ and its falsity as the negation of this:

$$f_{\rightarrow_2}((t_1, f_1), (t_2, f_2)) = (min(t_1 \ominus t_2, f_2 \ominus f_1), 1 - min(t_1 \ominus t_2, f_2 \ominus f_1))$$

The output of this conditional is obviously within the line of well-defined values and agrees with the Łukasiewicz conditional for this range of truth values. Actually, when restricted to the four truth values of FDE, it gives the semantics for an implication connective put forward by Smiley[9]. The truth table for this implication connective is as follows:

$\rightarrow$	1	b	n	0
1	1	0	0	0
b	1	1	0	0
n	1	0	1	0
0	1	1	1	1

Conditional 3

A final possibility for defining a conditional, which I am prompted to consider particularly for use as a conditional for a fuzzy relevant logic I discuss in the last section of this paper, is as follows:

$$f_{\rightarrow_3}((t_1, f_1), (t_2, f_2)) = (t_1 \ominus t_2, t_1 * f_2)$$

This third conditional agrees with the Łukasiewicz conditional, as when confined to the well-defined truth values, $t_1 * f_2 = 1 - (t_1 \ominus t_2)$.

When restricted to the values 1 (1,0), b (1,1), n (0,0) and 0 (0,1), the truth conditions for this connective and corresponding truth table are as follows:

$v(A \rightarrow_3 B) = (1, f)$ iff (if $v(A) = (1, f)$ then $v(B) = (1, f)$), $[f \in \{0, 1\}]$

$v(A \rightarrow_3 B) = (t, 1)$ iff ($v(A) = (1, f)$ and $v(B) = (t, 1)$), $[t, f \in \{0, 1\}]$

[9]See [1] and [18]

$\to$	1	b	n	0
1	1	b	n	0
b	1	b	n	0
n	1	1	1	1
0	1	1	1	1

A conditional with this truth table has been suggested as the conditional for a formal framework capturing situation semantics, something which will be discussed later on.

A brief comparison of the logics obtained by adding these three conditionals to FDE_∞ is given in Appendix B.

Quantification

The addition of universal and existential quantifiers to FDE_∞ is straightforward, the basic idea being to treat universal quantification as extended conjunction and existential quantification as extended disjunction. Where D is the domain of quantification and $A_x(d)$ the result of substituting d for x

$$v(\forall xA) = (t, f), \text{ where } t = Glb(\{v_t(A_x(d))|d \in D\}),\ f = Lub(\{v_f(A_x(d))|d \in D\})$$

$$v(\exists xA) = (t, f), \text{ where } t = Lub(\{v_t(A_x(d))|d \in D\}),\ f = Glb(\{v_f(A_x(d))|d \in D\})$$

Following are two small examples

	Q
a	(0.5,0.8)
b	(0.8,0)
c	(0.3,0.7)

With respect to the above model, $v(\forall xQ(x)) = (0.3, 0.8)$ and $v(\exists xP(x)) = (0.8, 0)$

P	a	b	c
a	(0.3,0.1)	(0.9,0.5)	(0.9,1)
b	(0.7,0.6)	(0.4,0.2)	(1,0.9)
c	(1,0.7)	(0.5,0.2)	(1,0.3)

With respect to the above model, $v(\forall x\forall yP(x, y)) = (0.3, 1)$ and $v(\exists x\forall yP(x, y)) = (0.5, 0.7)$, $v(\exists x\exists yP(x, y)) = (1, 0.1)$ and $v(\forall x\exists yP(x, y)) = (0.9, 0.2)$

Applications

I shall now look at applying the central ideas of this paper and sketch some possible applications of the type of formal system for which FDE_∞ is a basis. Similar motivations in going from classical logic to L or FDE can be seen in going from L or FDE to FDE_∞.

Philosophical Issues

Fuzzy Self-Reference

A venerable argument for the existence of truth-value gluts is the so called *Liar Paradox.* There are a number of paradoxes of the Liar family, the simplest example being the sentence 'This sentence is false', which must be false if it is true, and true if it is false; in either case it is both true and false. A standard way in which a logic can formally capture this fact whilst tolerating the contradictory conjunction of the liar sentence and its negation is by providing a way to evaluate the conjunction of a statement and its negation to a designated value. In doing so, the classical principle that anything follows from contradictory premises is evaded. One way of achieving this in with a *fuzzy FDE* is by assigning the liar sentence, p, the glutty truth value $(1, 1)$, so that $v(p \wedge \neg p) = (1, 1)$. Another way is to set the value of δ to be 0.5 and assigning p the truth value $(0.5, 0.5)$, so that $v(p \wedge \neg p) = (0.5, 0.5)$. Of course, the first strategy is inherited from *FDE* and the second strategy is inherited from *Ł*.

The availability of fuzzy over-defined truth values in *fuzzy FDE* provides a way to accommodate 'fuzzy liars', cases where liar sentences are given intermediate truth values between 0 and 1. Say, if the sentence 'This sentence is false' is 0.75 true, then it is 0.75 false, and vice-versa. It seems that the truth value of such a sentence might be suitably represented by the truth-value tuple (0.75, 0.75).

Fuzzy non-well-defined truth values seem applicable to other related examples, such as the following variation of Yablo's Paradox[10]:

y_1 for all $k > 1$, y is 1/1 untrue,
y_2 for all $k > 2$, y is 1/2 untrue,
y_3 for all $k > 3$, y is 1/3 untrue,
$\vdots$

These cursory thoughts would benefit from a rigorous analysis.

Hybrid Sentences

Fuzzy FDE also provides a formal framework for the evaluation of compound sentences comprised of sentences which have fuzzy truth values and sentences which have non-well-defined truth values. For example, consider the following two sentences

1. The word 'heterological' is itself heterological

2. John is bald

[10]See [19]

Heterological words do not apply to themselves. The word 'short' is short, so refers to itself and is therefore not heterological (autological). The word 'apple' is not an apple, so it does not refer to itself and it therefore is heterological. What of the word 'heterological'? If it does not refer to itself then it does, and if it does refer to itself then it does not. Sentence 1 is both true and false so an appropriate truth value for it is (1,1). As for the second sentence, take John to be a middle-aged man who has lost a significant amount of hair and is not thoroughly hirsute although is not totally bald. Say the sentence asserting his baldness has the fuzzy truth value (0.6,0.4). Given both an over-defined sentence and a well-defined yet fuzzy sentence such as these, within a *fuzzy FDE* the simple compound sentence conjoining these two sentences has a truth value of (min(1,0.6), max(1,0.4)), which equates to (0.6,1), a fuzzy-over-defined truth value.

Sorites Progressions Gaps

The sorites paradox is the name given to a class of paradoxical arguments related to gradual or continuous change, which arise as a result of indeterminate application of the vague predicates involved, vague predicates such as heap, bald and red.[11] A classic formulation of the paradox concludes that no amount of sand constitutes a heap. This is because (1) a single grain of sand does not constitute a heap and (2) if n grains of sand do not make a heap, then $n + 1$ grains do not make a heap. Therefore, by inductive reasoning, no number of grains of sand yields a heap. So 'heap' is a vague predicate.

One response to this paradox involves the employment of a fuzzy logic such as Ł because of its suitability for covering the existence of a relatively continuous change along a sorites progression and the fact that a failure of *modus ponens* in Ł given a suitable set of designated values invalidates the sorites chain of reasoning.

Another suggestion is that vagueness requires us to reject a simple dichotomy between truth and falsity. Just as a vague predicate divides objects into the positive extension, negative extension and the penumbra, vague sentences can be divided into the true, the false and the indeterminate. Adopting a suitable three-valued logic such as K_3, one can have it so that there is some i, such that S_i is true and S_{i+1} is neither true nor false, therefore $S_i \supset S_{i+1}$ is not true. Thus in denying some premises of the sorites argument, it fails.[12]

A not unexpected concern with this approach, of using a third truth value representing a truth-value gap, is that the notion, in a sorites progression, of a sharp boundary between truth and indeterminacy or indeterminacy and falsity is just as problematic as that between truth and falsity.

Within a *fuzzy FDE* framework, this issue of discreteness can be addressed whilst retaining an appeal to truth-value gaps by using the continuous boundary between true and neither true nor false and between neither true nor false and false. The relatively continuous change along a sorites progression can be modeled with a sequence of truth values such as (1,0), (0.9,0), (0.8,0) ... (0,0) ... (0,0.8), (0,0.9), (0,1), illustrated in figure 10.

[11]Dominic Hyde, 'Sorites Paradox', *The Stanford Encyclopedia of Philosophy* (Fall 2005 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2005/entries/sorites-paradox/>.

[12]Priest, 2001, p. 213.

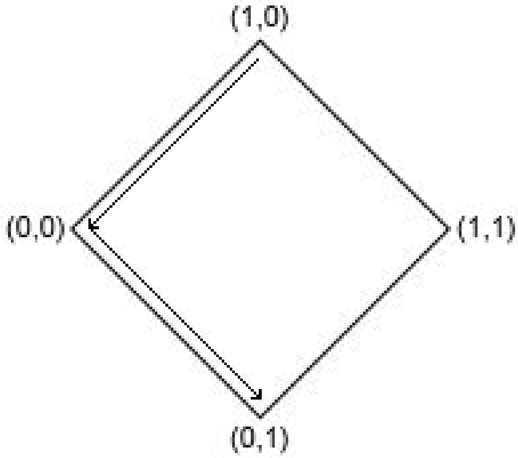

Figure 10: Progression of truth values in a gappy Sorites Paradox

Fuzzy Situation Semantics

Situation semantics is an approach to natural language semantics which is based on the idea that "sentences stand for facts or something like them"[13]. The project was initiated by the work of Barwise and Perry.[14]

In situation semantics, states of affairs, or *infons* as they are known in technical parlance, are complexes of properties and objects which are used to represent facts. Situations are limited parts or aspects of reality, which determine whether or not a state of affairs is supported. Consider the question of whether or not a footballer sustained an injury at a certain time T. There are two possible states of affairs, that they did or did not. The situation at T, on the field where the footballer was at the time, determines which of these states of affairs is the case.[15]

Situation theory, the formal theory that underlies situation semantics, formalises the nature of the supports relation. Information is always taken to be information about some situation and is built up from discrete informational items, the *infons* mentioned earlier. Infons are of the form

$$\langle\langle R, a_1, ..., a_n, 1\rangle\rangle, \langle\langle R, a_1, ..., a_n, 0\rangle\rangle$$

Infons are used to carry the information conveyed by a statement. They are not things that in themselves are true or false. Rather, infons may be true or false with regard to a certain situation. Given a situation, s, and an infon δ, we write

$$s \models \delta$$

[13] John R. Perry. (1998). 'Semantics, situation'. In E. Craig (Ed.), *Routledge Encyclopedia of Philosophy.* London: Routledge. Retrieved May 28, 2006, from http://www.rep.routledge.com/article/U041SECT2

[14] See [2]

[15] Perry, 1998.

to indicate that the infon δ is made factual, or is supported, by the situation s. Thus,

$$s \vDash \langle\langle R, a_1, ..., a_n, 1\rangle\rangle$$

means that, in situation s, the objects $a_1...a_n$ stand in the relation R, and

$$s \vDash \langle\langle R, a_1, ..., a_n, 0\rangle\rangle$$

means that in situation s, the objects $a_1...a_n$ do not stand in the relation R.[16]

To illustrate, consider a simple situation; the events on the football field mentioned above. The bits of information constituting the state of affairs in question are the footballer, the property of sustaining the injury and the time.

There are two possibilities or 'states of affairs', corresponding to whether the footballer injured themselves or not, which can be represented as:

$$\sigma : \langle\langle \text{ sustains injury, t, footballer, 1 } \rangle\rangle$$

and

$$\sigma' : \langle\langle \text{ sustains injury, t, footballer, 0 } \rangle\rangle$$

Let s be the situation on the football field. Then,

$$s \vDash \sigma$$

that is, s supports σ, or s makes it the case that σ.

Infons may be combined, recursively, using the operations of conjunction, disjunction, and situation-bounded existential and universal quantification to form compound infons.

Juan Barba Escriba has presented a formal system intended to capture some of the basic features of situation semantics, the details of which I shall not go into here.[17] Suffice it to say, the system includes formal counterparts of situations, including partial and incoherent ones, so that an infon can be supported and rejected simultaneously by the same situation. This naturally results in a quantified four valued logic, with the truth values 'true', 'false', 'true and false' and 'undefined'. He defines several connectives, including $\neg$, $\wedge$ and $\rightarrow$. This fragment of the language is actually *FDE* + $\rightarrow_3$.

Given this and the notion of a *fuzzy FDE* outlined in this paper, a natural step is to use these ideas to develop a formal system for a *fuzzy situation semantics*. Particularly in more complex situations, where the determination of a state of affairs by a situation is not epistemically clear-cut, this would be the way to go.

A suitable representation for the corresponding support relation is $\vDash_\delta$, where $\delta \in [0, 1]$, so that

[16]Keith Devlin, 'Situation Theory and Situation Semantics', accessed at http://www.stanford.edu/ kdevlin/HHL_SituationTheory.pdf

[17]Juan Barba Escriba, 'Two Formal Systems for Situation Semantics', *Notre Dame Journal of Formal Logic*, Vol. 33(1), 1992, pp. 70-88.

$$\sigma \models_\delta \langle\langle R, a_1, ..., a_n, 1\rangle\rangle$$

means that in situation s, the objects $a_1...a_n$ stand in the relation R by degree δ and

$$\sigma \models_\delta \langle\langle R, a_1, ..., a_n, 0\rangle\rangle$$

means that in situation s, the objects $a_1...a_n$ does not stand in the relation R by degree δ.

Evidence Logic

In seeking a logic for the provision of a knowledge representation framework, it is natural to look beyond the limitations of classical logic. Whereas an agent's knowledge of the real world is gradational, classical logic is absolute; Whereas an agent's knowledge of the real world is both confirmatory and refutatorily evidential, classical logic is simply confirmatory. Furthermore, since intermediate levels of conflict between confirmatory and refutatory evidence often arise, a reasonable paraconsistent framework for the processing of such conflict is desirable.[18]

Evidence Logic[19] accommodates this epistemic imperfection, providing a framework for the representation of both confirmatory and refutatory predication, with levels of evidential support. Precisely, for each $n > 1$ the Evidence Logic EL_n is defined as follows. Let

$$E_n = \{i/(n-1) : i = 1, ..., n-1\}$$

be the Evidence Space of size $n-1$. The evidence spaces E_n are used in Evidence Logic to provide measures of 'evidence levels'. In general, in applications, n shall have to be chosen sufficiently large to handle the grades of evidence, whilst a practical bound to n will be dictated by the situation. For each s-ary predicate symbol P, and for any terms $t_1, ..., t_s$ and any e in E_n, EL_n contains atomic formulas

$$P_c(t_1, ..., t_s) : e \text{ and } P_r(t_1, ..., t_s) : e$$

where the former asserts that there is evidence at level e confirming $P(t_1, ..., t_s)$ while the latter asserts evidence at level e refuting $P(t_1, ..., t_s)$.[20].

The similarities between the ideas of Evidence Logic and FDE_n are evident. The Evidence Space of EL_{n+1} corresponds in size to the set $\mathcal{T}$ in the logic FDE_n. The Evidence Logic formulas $P_c(t_1, ..., t_s) : e$ and $P_r(t_1, ..., t_s) : e_r$, can be translated to the FDE_n valuation $v(P(t_1, ..., t_s)) = (e_c, e_r)$. Formalisation and study of Evidence Logic concepts from an FDE_n perspective might be worthwhile. For an overview of work on Evidence Logic, consult [7].

[18] Don Faust. 'Between Consistency and Paraconsistency: Perspectives from Evidence Logic', in Carinielli, Walter A. Cogniglio, Marcelo E. and Itala M. Loffredo D'Ottaviano (eds)., *Paraconsistency the logical way to the inconsistent*, New York, Marcel Dekker, 2002, p. 501.

[19] See [7]

[20] Faust, p. 501.

Computing

Both fuzzy logic and bilattices such as *FDE* have been suggested or implemented as information processing frameworks.

Fuzzy computing, that is, the use of fuzzy logic as a logic for information processing systems has gained widespread use. Employment of fuzzy logic is one of the techniques of soft-computing, the use of "computational methods tolerant to suboptimality and impreciseness (vagueness) and giving quick, simple and sufficiently good solutions".[21] Other computing applications of fuzzy logic were discussed earlier.

In some seminal papers[22], Nuel Belnap suggested *FDE* be used as a basis for computer programming semantics, since it allows for an elegant dealing of missing or conflicting information. He remarks

> what is the computer to do [with inconsistent information]? If it is a classical two-valued logician, it must give up altogether talking about anything to anybody or, equivalently, it must say everything to everybody. ... [In cases where] there is a possibility of inconsistency, we want to set things up so that the computer can continue reasoning in a sensible manner even if there is such an inconsistency, discovered or not.[23]

Belnap outlines an information processing system which when given information asserting an item marks that item with a 'told True' and when given information denying an item marks that item with a 'told False'. Each item in the database will therefore be marked in one of the following ways:[24]

1. just the "told True" sign, indicating that that item has been asserted to the computer without ever having been denied (***T***)
2. just the value "told False", which indicates that the item has been denied but never asserted (***F***)
3. No "told" values at all, which means the computer is in ignorance, has been told nothing (***None***)
4. The interesting case: the item might be marked with both "told True" and "told False" (***Both***)

In Belnap's simple example, after one person enters information concerning the results of a baseball game into the system it looks something like this:

⟨ Pirates, 1971 ⟩ True

⟨ Orioles, 1971 ⟩ False

[21] Hájek, 2002
[22] See [3] and [4]
[23] Nuel Belnap, 'A Useful Four-Valued Logic', in J. Michael Dunn and George Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, Dordrecht 1977, p. 9.
[24] *Ibid.*, p. 11.

Subsequently another person feeds the system incorrect information, after which the first entry looks something like this:

$\langle Pirates, 1971\rangle$ True, False

In such an application, the four values are "unabashedly epistemic"[25], used to represent the system's evaluation of a state of affairs.

Given the promise of both of these computing paradigms, an obvious move is to combine the two for an information processing system that is both sensitive to degrees of information and missing or conflicting information. I will briefly outline a few ideas.

In an epistemic context, the four values of *FDE* can be considered as evaluating two aspects of a state of affairs related to a database:

1. whether there is positive information about the truth of this state of affairs $(1, f)$ or not $(0, f)$
2. whether there is positive information about the falsity of this state of affairs $(t, 1)$ or not $(t, 0)$

Translating this to *fuzzy FDE*, the range of truth values can be considered as evaluating:

1. the *degree* to which there is positive information about the truth of this state of affairs
2. the *degree* to which there is positive information about the falsity of this state of affairs

Belnap suggests something like this in passing:

> My penultimate observation concerns the suggestion that the computer keep more information than I have allowed it to keep. Perhaps it should count the number of times it has been told True or told False ... [but] it is by no means self-evident how this extra information is to be utilised in answering question, in inference, and in the input of complex sentences.[26]

One suitable option to address this observation would be the employment of something like the evidence logic described in the previous section, where each inputted assertion of an item adds a degree of positive information about its truth and each inputted denial of the item adds a degree of positive information about its falsity.

Another possibility is to have a system which accepts well-defined, yet fuzzy inputs. For example, one person could assert an item with degree of truth 0.7 and another person could assert an item with degree of truth 0.2, resulting in the item being marked with a value of (0.7,0.8); each item of input the system receives is well-defined yet fuzzy and the aggregate of input it receives might be inconsistent. If it is to tolerate and accurately represent the inconsistent data it holds, it will need to use fuzzy non-well-defined values.

[25] Nuel Belnap, 'How a computer should think', in G.Ryle, *Contemporary Aspects of Philosophy*, London, 1975, p. 43.

[26] *Ibid.*, p. 49.

Finally, fuzzy non-well-defined truth values can be used for output in an inconsistency tolerant question answering system.

Take a database consisting of the following consistent information:

shorter(tom, dick).
shorter(tom, harry).
shorter(dick, harry).
taller(dick, tom).
taller(harry, tom).
taller(harry, dick.

A simple program could be written which calculates a truth value, (t, f), for each question of the form *short(x)?* and each question of the form *tall(x)?* ($\neg$ *short(x)?*), where $t, f \in \{i/(n-1)|0 \leq i \leq n-1\}$ and n is the number of domain elements (in this case 3). Furthermore, the program works towards uniquely pairing each instance of the statement *short(x)* with one of the truth values and uniquely pairing each instance of the statement *tall(x)* with one of the truth values.

Following is some pseudocode for such a program:

```
for each member of the domain
    set shortness of member to 0
for each member of the domain
    set tallness of member to 0

for i = 0 to i = number of domain elements
    add i/(n-1) to the set of truth degree values

set increment value to lowest truth degree value that is not 0

for each item of information in the database of the form shorter(x,y)
    add increment value to shortness of x

for each item of information in the database of the form taller(x,y)
    add increment value to tallness of x
```

The output of such a program given this database can be summed up in the following table:

	shortness	tallness
tom	(1, 0)	(0, 1)
dick	(0.5, 0.5)	(0.5, 0.5)
harry	(0, 1)	(1, 0)

If however, the database contained inconsistent information, so that say, *taller(dick, tom)* was replaced by *taller(tom, dick)*, not every statement of the form *short(x)* or *tall(x)* will be assigned a well-defined truth value. The output of this program in such a case would look something like this:

	shortness	tallness
tom	(1, 0)	(0.5, 1)
dick	(0.5, 0.5)	(0, 0.5)
harry	(0, 1)	(1, 0)

Of course, this program is just a cursory example and a practicable application of this idea would require a much more sophisticated program.

An implication connective and consequence relation for such applications is something which remains to be looked into. Belnap discusses these two aspects and remarks that

> the inference from A to B is valid, or A entails B, if the inference never leads us from the True to the absence of the True (preserves Truth), and also never leads us from the absence of the False to the False (preserves non-Falsity).[27]

This consideration suggests that $\rightarrow_1$ or more simply yet adequately, $\rightarrow_2$ are suitable. It also suggests the possibility of considering a consequence relation which differs to that I described earlier.[28]

Fuzzy Relevant Logic

Before closing, I would like to look at the idea of constructing a fuzzy relevant logic based on FDE_∞.[29] FDE is used as a basis for relevant logics and as this paper has shown, the relational semantics for FDE can be fuzzified, resulting in a fuzzy logic with independent degrees of truth and falsity. Given this, we have a strategy for constructing a logic that is both fuzzy and relevant, by combining FDE_∞ with a relevant logic based on relational semantics. I will outline this idea using the basic relevant logic N_4.[30]

The logic N_4 is a structure $\langle W, N, v\rangle$, where W is a set of worlds, $N \subseteq W$ is the set of normal worlds and v is a function[31] which does two things. Firstly it assigns a truth value tuple (t, f), where $t, f \in \{0, 1\}$ to each pair comprising a world, $w \in W$, and a proposition, p. This is written as $v_w(p) = (t, f)$. Secondly, for every non-normal world, $w \in W - N$, v assigns a truth value tuple (t, f) to formulas of the form $A \rightarrow B$.

The truth and falsity conditions for the extensional connectives, $\wedge$, $\vee$ and $\neg$ are the same as those for FDE, although relativised to each world. The truth and falsity conditions for $\rightarrow$ are:

$$v_w(A \rightarrow B) = (1, f) \text{ iff for all } w' \in W \text{ such that } v'_w(A) = (1, f),\ v'_w(B) = (1, f).\ [f \in \{0, 1\}]$$

[27] Belnap, 1975, p.43

[28] This is something that did not come to my mind, though it might technically not make a difference. Belnap writes “I note that Dunn 1975 has shown that it suffices to mention truth-preservation, since if some inference form fails to always preserve non-Falsity, then it can be shown by a technical argument that it also fails to preserve Truth”. See [3], p. 43. What exactly this means I have not as yet looked into.

[29] Graham Priest outlines strategies for constructing fuzzy relevant logics in [14] and [16]. The approach I outline differs in that the set of fuzzy values extends beyond the set of well-defined fuzzy values. Whether this provides an advantage for intended applications is an open matter.

[30] See [14], Chapter 9

[31] Comprised of subfunctions v_t and v_f like in section 3

$v_w(A \rightarrow B) = (t, 1)$ iff for some $w' \in W$ such that $v'_w(A) = (1, f)$, $v'_w(B) = (t, 1)$. $[t, f \in \{0, 1\}]$

Combining this basic relevant logic with FDE_∞ results in a new type of fuzzy relevant logic, which I will here refer to as N_∞. The conditions for $\wedge$, $\vee$ and $\neg$ in this logic are the same as those for FDE_∞. For the truth and falsity conditions for $\rightarrow$, I refer back to the conditional operator $\rightarrow_3$ I defined in section 4.

$v_w(A \rightarrow B) = (t, f)$, where $t = Glb(\{v_{t_{w'}}(A \rightarrow_3 B) | w' \in W\})$ and $f = Lub(\{v_{f_{w'}}(A \rightarrow_3 B) | w' \in W\})$

The consequence relation of this logic is defined as follows:

$\Sigma \vDash A$ iff for every interpretation, $\langle W, N, v\rangle$, and all $w \in W$, $Glb(v_t[\Sigma]) \leq v_t(A)$[32]

Since every N_4-interpretation is a N_∞-interpretation where every formula takes the value $(1, 0), (1, 1), (0, 0)$ or $(0, 1)$, N_∞ is a sub-logic of N_4: if $\Sigma \vDash_{N_\infty} A$ then $\Sigma \vDash_{N_4} A$. Hence N_∞ is a relevant logic.[33]

Conclusion

This paper has outlined a merging of the ideas inherent in *fuzzy logic* and *first degree entailment*. The range of subtopics and issues covered can no doubt be further investigated. Looking further into the merits and suitability of the three conditionals I defined as well as comparing the logics they generate is one place to start. I discussed a few ways in which the under-defined and over-defined values of this type of logic as well as the connectives can be used to deal with certain fuzzified philosophical problems involving truth-value gluts/gaps. There might be richer examples of paradoxes combining fuzzy concepts such as vagueness with glutty or gappy concepts like self-reference which would contribute to proving the worth of the ideas in this paper. The worth of these ideas is manifest with regard to their application to information processing tasks and agent programming, however work on formalising this application remains to be done. Finally, there remains a range of meta-theoretical properties to investigate. Two of the most prominent are the development of proof systems for the logics I have discussed, in particular $BN4_\infty$ and a systematic examination of the relationships between the members of the $BN4_n$ family of logics.

[32] $v_t[\Sigma]$ is explained in section 3

[33] relevant iff whenever $A \rightarrow B$ is logically valid, A and B have a propositional parameter in common. See [14], Chapter 9.

Appendix A

Verification that $\rightarrow_1$ gives the *BN4* conditional:

$f_{\rightarrow_1}((1,0),(1,0)) = (min(min(1, 1-1+1), min(1, 1-0+0)), max(0, 1+0-1)) = (1,0)$ [1]
$f_{\rightarrow_1}((1,0),(1,1)) = (min(min(1, 1-1+1), min(1, 1-1+0)), max(0, 1+1-1)) = (1,0)$ [0]
$f_{\rightarrow_1}((1,0),(0,0)) = (min(min(1, 1-1+0), min(1, 1-0+0)), max(0, 1+0-1)) = (0,1)$ [n]
$f_{\rightarrow_1}((1,0),(0,1)) = (min(min(1, 1-1+0), min(1, 1-1+0)), max(0, 1+1-1)) = (0,0)$ [0]
$f_{\rightarrow_1}((1,1),(1,0)) = (min(min(1, 1-1+1), min(1, 1-0+1)), max(0, 1+0-1)) = (0,1)$ [1]
$f_{\rightarrow_1}((1,1),(1,1)) = (min(min(1, 1-1+1), min(1, 1-1+1)), max(0, 1+1-1)) = (1,0)$ [b]
$f_{\rightarrow_1}((1,1),(0,0)) = (min(min(1, 1-1+0), min(1, 1-0+1)), max(0, 1+0-1)) = (1,1)$ [n]
$f_{\rightarrow_1}((1,1),(0,1)) = (min(min(1, 1-1+0), min(1, 1-1+1)), max(0, 1+1-1)) = (0,0)$ [0]
$f_{\rightarrow_1}((0,0),(1,0)) = (min(min(1, 1-0+1), min(1, 1-0+0)), max(0, 0+0-1)) = (0,1)$ [1]
$f_{\rightarrow_1}((0,0),(1,1)) = (min(min(1, 1-0+1), min(1, 1-1+0)), max(0, 0+1-1)) = (0,0)$ [n]
$f_{\rightarrow_1}((0,0),(0,0)) = (min(min(1, 1-0+0), min(1, 1-0+0)), max(0, 0+0-1)) = (1,0)$ [1]
$f_{\rightarrow_1}((0,0),(0,1)) = (min(min(1, 1-0+0), min(1, 1-1+0)), max(0, 0+1-1)) = (0,0)$ [n]
$f_{\rightarrow_1}((0,1),(1,0)) = (min(min(1, 1-0+1), min(1, 1-0+1)), max(0, 0+0-1)) = (1,0)$ [1]
$f_{\rightarrow_1}((0,1),(1,1)) = (min(min(1, 1-0+1), min(1, 1-1+1)), max(0, 0+1-1)) = (1,0)$ [1]
$f_{\rightarrow_1}((0,1),(0,0)) = (min(min(1, 1-0+0), min(1, 1-0+1)), max(0, 0+0-1)) = (1,0)$ [1]
$f_{\rightarrow_1}((0,1),(0,1)) = (min(min(1, 1-0+0), min(1, 1-1+1)), max(0, 0+1-1)) = (1,0)$ [1]

Proof that $\rightarrow_1$ gives the Łukasiewicz conditional:

Recall that as an operation on truth value tuples, $\ominus$ can be defined as:

$$f_{\rightarrow}((t_1, f_1), (t_2, f_2)) = (t_1 \ominus t_2, 1 - t_1 \ominus t_2)$$

and $\rightarrow_1$ can be defined as

$$f_{\rightarrow}((t_1, f_1), (t_2, f_2)) = (min(t_1 \ominus t_2, f_2 \ominus f_1), t_1 * f_2)$$

so, when confined to truth values (t, f), where $t, f \in [0, 1]$ and $t + f = 1$, it should be the case that

1. $min(t_1 \ominus t_2, f_2 \ominus f_1) = t_1 \ominus t_2$

   and

2. $t_1 * f_2 = 1 - t_1 \ominus t_2$

Proof of 1:
$min(t_1 \ominus t_2, f_2 \ominus f_1) = t_1 \ominus t_2$
$\Rightarrow min(min(1, 1 - t_1 + t_2), min(1, 1 - f_2 + f_1)) = min(1, 1 - t_1 + t_2)$
$f_1 = 1 - t_1$ and $f_2 = 1 - t_2$
$\Rightarrow min(min(1, 1 - t_1 + t_2), min(1, 1 - (1 - t_2) + (1 - t_1))) = min(1, 1 - t_1 + t_2)$
$\Rightarrow min(min(1, 1 - t_1 + t_2), min(1, 1 - t_2 + t_1)) = min(1, 1 - t_1 + t_2)$

$\Rightarrow min(min(1, 1 - t_1 + t_2), min(1, 1 - t_1 + t_2)) = min(1, 1 - t_1 + t_2)$
$\Rightarrow min(1, 1 - t_1 + t_2) = min(1, 1 - f_2 + f_1)$
$\Rightarrow 1 - t_1 + t_2 = 1 - f_2 + f_1$
$\Rightarrow 1 - t_1 + t_2 + f_2 = 1 + f_1$
$\Rightarrow 1 - t_1 + 1 = 1 + f_1$
$\Rightarrow 2 - t_1 = 1 + f_1$
$\Rightarrow 1 = t_1 + f_1$
$\Rightarrow 1 = 1$
$\therefore min(t_1 \ominus t_2, f_2 \ominus f_1) = t_1 \ominus t_2$

Proof of 2:
$t_1 * v(f_2) = 1 - t_1 \ominus t_2$
$\Rightarrow max(0, t_1 + f_2 - 1) = 1 - min(1, 1 - t_1 + t_2)$
$\Rightarrow max(0, t_1 + f_2 - 1) + min(1, 1 - t_1 + t_2) = 1$
$f_2 = 1 - t_2$
$\Rightarrow max(0, t_1 + (1 - t_2) - 1) + min(1, 1 - t_1 + t_2) = 1$
$\Rightarrow max(0, t_1 - t_2) + min(1, 1 - t_1 + t_2) = 1$
if $t_2 > t_1$ then $t_1 - t_2 < 0$ and $1 - t_1 + t_2 > 1$ so
$max(0, t_1 - t_2) + min(1, 1 - t_1 + t_2) = 0 + 1 = 1$
if $t_2 <= t_1$ then $max(0, t_1 - t_2) = t_1 - t_2$ and $min(1, 1 - t_1 + t_2) = 1 - t_1 + t_2$ so
$\Rightarrow t_1 - t_2 + 1 - t_1 + t_2 = 1$
$\Rightarrow 1 = 1$
$\therefore t_1 * f_2 = 1 - t_1 \ominus t_2$

Appendix B

Let $FDE_\infty + \to_n$ denote the logic resulting from adding one of the 3 conditionals to FDE_∞. It can be seen that for no $x, y \in \{1, 2, 3\}, x \neq y$ does the following hold: if $\Sigma \vDash_{FDE_\infty + \to_x} A$ then $\Sigma \vDash_{FDE_\infty + \to_y} A$

	$\to_1$	$\to_2$	$\to_3$
(1) $q \vDash p \to q$	$\times$	$\times$	$\surd$
(2) $\neg p \vDash p \to q$	$\times$	$\times$	$\times$
(3) $(p \wedge q) \to r \vDash (p \to r) \vee (q \to r)$	$\surd$	$\surd$	$\surd$
(4) $\neg(p \to p) \vDash q$	$\times$	$\surd$	$\times$
(5) $\neg(p \to q) \vDash p$	$\surd$	$\times$	$\surd$
(6) $p \to r \vDash (p \wedge q) \to r$	$\surd$	$\surd$	$\surd$
(7) $p \to q, q \to r \vDash p \to r$	$\surd$	$\surd$	$\surd$
(8) $p \to q \vDash \neg q \to \neg p$	$\surd$	$\surd$	$\times$
(9) $\vDash p \to (q \vee \neg q)$	$\times$	$\times$	$\times$
(10) $\vDash (p \wedge \neg p) \to q$	$\times$	$\times$	$\times$

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