# More on Dual-Intuitionistic Logic

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**Abstract**

I begin this paper by defining a notion of duality which holds in classical logic. I then consider a sequent system which reverses the restriction typical of sequent systems for intuitionistic logic; where $A$ is a formula and $X$ is a set of formulas, rather than sequents of the form $X \vdash A$, sequents are of the form $A \vdash X$. This sequent system is thus *dual* to its corresponding intuitionistic sequent system in the sense of my earlier definition and defines what Urbas [19] has called *dual-intuitionist logic*. I survey the main results for this logic established in [19] before going on to prove a few more results. I then consider two Kripke semantics for dual-intuitionist logic and briefly discuss its suitability as a logical framework for scientific method.

## 1 Defining the Duality and Dual-Intuitionistic Logic

Genzten introduced his calculi **LK** and **LJ** as formalisms for classical logic and intuitionistic logic respectively. Since then Gentzen calculi or sequent systems have become a most popular proof-theoretic method. There are a variety of equivalent sequent systems for classical logic, here is one them.[1]

$$\text{Identity and Cut} \quad A \vdash A\ [ID] \quad \frac{X \vdash Y, C \qquad C, X' \vdash Y'}{X, X' \vdash Y, Y'}\ Cut$$

$$\text{Conditional Rules} \quad \frac{X \vdash Y, A \qquad B, X' \vdash Y'}{X, X', A \to B \vdash Y, Y'} \to L \quad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y} \to R$$

$$\text{Negation Rules} \quad \frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg L \quad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg R$$

$$\text{Conjunction Rules} \quad \frac{X, A \vdash Y}{X, A \wedge B \vdash Y}\ \wedge L_1 \quad \frac{X, A \vdash Y}{X, B \wedge A \vdash Y}\ \wedge L_1 \quad \frac{X \vdash A, Y \qquad X' \vdash B, Y'}{X, X' \vdash A \wedge B, Y, Y'}\ \wedge R$$

$$\text{Disjunction Rules} \quad \frac{X, A \vdash Y \qquad X, B \vdash Y}{X, A \vee B \vdash Y}\ \vee L \quad \frac{X \vdash A, Y}{X \vdash A \vee B, Y}\ \vee R_1 \quad \frac{X \vdash A, Y}{X \vdash B \vee A, Y}\ \vee R_2$$

$$\text{Structural Rules} \quad \frac{X, A, A \vdash Y}{X, A \vdash Y}\ WL \quad \frac{X \vdash A, A, Y}{X \vdash A, Y}\ WR \quad \frac{X \vdash Y}{X, A \vdash Y}\ KL \quad \frac{X \vdash Y}{X \vdash A, Y}\ KR$$

Figure 1: Sequent rules for classical logic

Since the rules in Figure 1 characterise classical logic, the sequents $A \to B \vdash \neg A \vee B$ and $\neg A \vee B \vdash A \to B$ are derivable. To these rules can be added the rules for another connective, which I shall denote with the $\doteq$ symbol and call "subtraction".[2]

$$\text{Subtraction Rules} \quad \frac{A, X \vdash Y, B}{A \doteq B, X \vdash Y} \doteq L \quad \frac{X \vdash A, Y \qquad X', B \vdash Y'}{X, X' \vdash Y, Y', A \doteq B} \doteq R$$

Figure 2: Subtraction rules for classical logic

[1] Throughout this paper for sequents I will use $p, q, ...$ to denote atomic formulas, $A$, $B$, $C$ and $D$ to denote single active formulas, $L$ and $R$ to denote no more than a single passive formula and $X, X', Y, Y'$ to denote non-specific multisets of formulas. The structural rule of interchanging formulas in a cedent is implicit. Also, theorems unaccompanied by proof will be taken as fact.

[2] This type of connective has been discussed in the literature, though technically in the context of logics other than classical logic, where as will be seen, its presence is more significant. Urbas [19] employs the symbol $\doteq$ to denote this type of connective and Restall [15] calls it 'subtraction', because in general $A \doteq B$ can be read as 'A without B' or 'A minus B'; hence my adoption of these terminological conventions.

**Theorem 1** *With classical logic,* $A \doteq B \equiv A \wedge \neg B$.

**Proof** The two following derivations

$$\dfrac{\dfrac{A \vdash A \qquad \dfrac{B \vdash B}{\vdash \neg B, B}\neg R}{A \vdash A \wedge \neg B, B}\wedge R}{A \doteq B \vdash A \wedge \neg B}\doteq L \qquad \dfrac{\dfrac{A \vdash A \qquad \dfrac{B \vdash B}{\neg B, B \vdash}\neg L}{A, \neg B \vdash A \doteq B}\doteq R}{A \wedge \neg B \vdash A \doteq B}\wedge L$$

■

Adding the following quantifier rules gives us classical predicate logic. $A[x/c]$ denotes the result of replacing the occurrences of $x$ in $A$ by $c$.

| | | |
|---|---|---|
| Universal Quantifier Rules | $\dfrac{X, A[x/c] \vdash Y}{X, (\forall x)A \vdash Y}\forall L$ | $\dfrac{X \vdash A[x/c], Y}{X \vdash (\forall x)A, Y}\forall R$ ($c$ not in $X, Y$) |
| Existential Quantifier Rules | $\dfrac{X, A[x/c] \vdash Y}{X, (\exists x)A \vdash Y}\exists L$ ($c$ not in $X, Y$) | $\dfrac{X \vdash A[x/c], Y}{X \vdash (\exists x)A, Y}\exists R$ |

Figure 3: Quantifier rules for classical logic

I will use **LK** to denote this classical sequent system.

**Definition 1** *The duality function* $^d$ *maps formulas to formulas, sets of formulas to sets of formulas and sequents to sequents as follows:*

$$\begin{array}{lcl}
p^d & = & p \\
(\neg A)^d & = & \neg(A)^d \\
(A \wedge B)^d & = & A^d \vee B^d \\
(A \vee B)^d & = & A^d \wedge B^d \\
(A \rightarrow B)^d & = & B^d \doteq A^d \\
(A \doteq B)^d & = & B^d \rightarrow A^d \\
((\forall x)Px)^d & = & (\exists x)(Px)^d \\
((\exists x)Px)^d & = & (\forall x)(Px)^d \\
\{A_1, ..., A_n\}^d & = & \{A_1^d, ..., A_n^d\} \\
(X \vdash Y)^d & = & Y^d \vdash X^d
\end{array}$$

**Theorem 2** *A sequent* $A \vdash B$ *is derivable if and only if its dual* $(A \vdash B)^d$ *is derivable. Furthermore, the dual of the derivation of* $A \vdash B$ *is a derivation of the dual of* $A \vdash B$.

**Remark** As mentioned above there are a variety of sequent systems for classical logic. One type of variation to that presented above involves replacing the two rules $\wedge L_1$ and $\wedge L_2$ / $\vee R_1$ and $\vee R_2$ with one rule $\wedge L$ / $\vee R$. For example:

$$\dfrac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\wedge L$$

Another less noted option would involve replacing the one rule $\rightarrow R$ / $\doteq L$ with two rules $\rightarrow R_1$ and $\rightarrow R_2$ / $\doteq L_1$ and $\doteq L_2$. I note this option as its significance will become made.

| | | |
|---|---|---|
| Alternative Subtraction Rules | $\dfrac{A, X \vdash Y}{A \doteq B, X \vdash Y}\doteq L_1$ | $\dfrac{X \vdash Y, B}{A \doteq B, X \vdash Y}\doteq L_2$ |
| Alternative Conditional Rules | $\dfrac{X, A \vdash Y}{X \vdash A \rightarrow B, Y}\rightarrow R_1$ | $\dfrac{X \vdash B, Y}{X \vdash A \rightarrow B, Y}\rightarrow R_2$ |

Figure 4: Alternative classical rules for subtraction and implication

Genzten's profound insight was that by imposing the restriction on the classical sequent system **LK** that succeedents may have at most one formula, what is obtained is the intuitionistic sequent system **LJ**, which characterises intuitionistic logic. Figure 5 contains one such system.

$$\text{Identity and Cut} \quad A \vdash A\ [ID] \quad \frac{X \vdash C \qquad C, X' \vdash R}{X, X' \vdash R}\ Cut$$

$$\text{Conditional Rules} \quad \frac{X \vdash A \qquad B, X' \vdash R}{A \to B, X, X' \vdash R} \to L \quad \frac{X, A \vdash B}{X \vdash A \to B} \to R$$

$$\text{Negation Rules} \quad \frac{X \vdash A}{X, \neg A \vdash} \neg L \quad \frac{X, A \vdash}{X \vdash \neg A} \neg R$$

$$\text{Conjunction Rules} \quad \frac{X, A \vdash R}{X, A \wedge B \vdash R} \wedge L_1 \quad \frac{X, A \vdash R}{X, B \wedge A \vdash R} \wedge L_2 \quad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B} \wedge R$$

$$\text{Disjunction Rules} \quad \frac{X, A \vdash R \qquad X, B \vdash R}{X, A \vee B \vdash R} \vee L \quad \frac{X \vdash A}{X \vdash A \vee B} \vee R_1 \quad \frac{X \vdash A}{X \vdash B \vee A} \vee R_2$$

$$\text{Structural Rules} \quad \frac{X, A, A \vdash R}{X, A \vdash R}\ WL \quad \frac{X \vdash R}{X, A \vdash R}\ KL \quad \frac{X \vdash}{X \vdash A}\ KR$$

$$\text{Universal Quantifier Rules} \quad \frac{X, A[x/c] \vdash R}{X, (\forall x)A \vdash R} \forall L \quad \frac{X \vdash A[x/c]}{X \vdash (\forall x)A} \forall R\ (c \text{ not in } X, Y)$$

$$\text{Existential Quantifier Rules} \quad \frac{X, A[x/c] \vdash R}{X, (\exists x)A \vdash R} \exists L\ (c \text{ not in } X, Y) \quad \frac{X \vdash A[x/c]}{X \vdash (\exists x)A} \exists R$$

Figure 5: Sequent rules for intuitionistic logic

Given all of this, a next step is to consider what sort of rules and connectives make sense in a sequent system with sequents of the form $A \vdash Y$, where $A$ is a single formula and $Y$ is a, possibly empty, multiset of formulas. What logic do we get if we impose a restriction on the set of classical sequent rules such that the antecedent, instead of the succeedent, can have at most one formula? Approaching this question, it helps to be mindful of the notion of duality given in Definition 1 (With an intuitionistic sequent, $X \vdash A$, which has multiple antecedents and at most one succeedent, its dual-intuitionistic sequent is $X^d \vdash A^d = A^d \vdash X^d$). Figure 6 contains sequent rules for such a logic, which shall be generally termed *dual-intuitionistic logic*.

$$\text{Identity and Cut} \quad A \vdash A\ [ID] \quad \frac{L \vdash C, Y \qquad C \vdash Y'}{L \vdash Y, Y'}\ Cut$$

$$\text{Negation Rules} \quad \frac{\vdash A, Y}{\neg A \vdash Y} \neg L \quad \frac{A \vdash Y}{\vdash \neg A, Y} \neg R$$

$$\text{Conjunction Rules} \quad \frac{A \vdash Y}{A \wedge B \vdash Y} \wedge L_1 \quad \frac{A \vdash Y}{B \wedge A \vdash Y} \wedge L_2 \quad \frac{L \vdash A, Y \qquad L \vdash B, Y'}{L \vdash A \wedge B, Y, Y'} \wedge R$$

$$\text{Disjunction Rules} \quad \frac{A \vdash Y \qquad B \vdash Y}{A \vee B \vdash Y} \vee L \quad \frac{L \vdash A, Y}{L \vdash A \vee B, Y} \vee R_1 \quad \frac{L \vdash A, Y}{L \vdash B \vee A, Y} \vee R_2$$

$$\text{Structural Rules} \quad \frac{L \vdash A, A, Y}{L \vdash A, Y}\ WR \quad \frac{\vdash Y}{A \vdash Y}\ KL \quad \frac{L \vdash Y}{L \vdash A, Y}\ KR$$

Figure 6: Sequent rules for dual-intuitionistic logic

The dualities listed in Definition 1 can be seen by comparing rules in Figure 6 with rules in Figure 5. For example, the rules for $\wedge$ in this dual-intuitionistic system are dual to the rules for $\vee$ in the intuitionistic system.

Defining a conditional for dual-intuitionistic logic is a little trickier than with intuitionistic logic. Using the standard classical rules for $\rightarrow$ and placing the restriction of singularity for the antecedent, we get a connective $\rightarrow'$:

$$\frac{\vdash Y, A \qquad B \vdash Y'}{A \rightarrow B \vdash Y, Y'} \rightarrow' L \qquad \frac{A \vdash B, Y}{\vdash A \rightarrow B, Y} \rightarrow' R$$

However, using the alternative classical rules for $\rightarrow R$ in Figure 4 and placing the restriction of singularity for the antecedent, we get a slightly different conditional, which is actually the conditional we want; what this means will become clearer. So to the dual-intuitionistic sequent rules of Figure 6 we add the following:

$$\frac{\vdash Y, A \qquad B \vdash Y'}{A \rightarrow B \vdash Y, Y'} \rightarrow L \qquad \frac{A \vdash Y}{\vdash A \rightarrow B, Y} \rightarrow R \qquad \frac{L \vdash B, Y}{L \vdash A \rightarrow B, Y} \rightarrow R$$

Figure 7: Conditional sequent rules for dual-intuitionistic logic

This observation of the difference between $\rightarrow$ and $\rightarrow'$ is made by Urbas [19], who explicates this distinction and demonstrates how $\rightarrow'$ can be derived from $\rightarrow$ but not vice-versa.

Quantifiers are added easily enough to dual-intuitionistic logic:

| | | |
|---|---|---|
| Universal Quantifier Rules | $\dfrac{A[x/c] \vdash Y}{(\forall x)A \vdash Y} \forall L$ | $\dfrac{L \vdash A[x/c], Y}{L \vdash (\forall x)A, Y} \forall R$ ($c$ not in $X, Y$) |
| Existential Quantifier Rules | $\dfrac{A[x/c] \vdash Y}{(\exists x)A \vdash Y} \exists L$ ($c$ not in $X, Y$) | $\dfrac{L \vdash A[x/c], Y}{L \vdash (\exists x)A, Y} \exists R$ |

Figure 8: Quantifier sequent rules for dual-intuitionistic logic

The resulting dual-intuitionistic system is denoted with **DLJ**.

Adding rules for the $\doteq$ connective is something we also want to consider for both intuitionistic and dual-intuitionistic logic. For dual-intuitionistic logic, we can use the standard classical sequent rules for $\doteq$ given in Figure 2 and impose the restriction of singularity for the antecedent.

$$\frac{A \vdash Y, B}{A \doteq B \vdash Y} \doteq L \qquad \frac{L \vdash A, Y \qquad B \vdash Y'}{L \vdash Y, Y', A \doteq B} \doteq R$$

Figure 9: Subtraction rules for dual-intuitionistic logic

Adding these rules to **DLJ** gives us the system denoted with $\mathbf{DLJ}^{\doteq}$. Similarly with **LK** and **LJ**

Analogously to the case of $\rightarrow$ for dual-intuitionistic logic, to incorporate the desired rules for $\doteq$ into intuitionistic logic, we should use the alternative classical rules for $\doteq L$ given in Figure 4 and place the restriction of singularity on the succeedent. The resulting system is denoted with $\mathbf{LJ}^{\doteq}$

$$\frac{A, X \vdash R}{A \doteq B, X \vdash R} \doteq L_1 \qquad \frac{X \vdash B}{A \doteq B, X \vdash} \doteq L_2 \qquad \frac{X \vdash A \qquad X', B \vdash}{X, X' \vdash A \doteq B} \doteq R$$

Figure 10: Subtraction rules for intuitionistic logic

It is apparent that the duality given in Definition 1 holds between intuitionistic logic and dual-intuitionistic logic:

**Theorem 3** *A sequent $A \vdash B$ is derivable in $\boldsymbol{DLJ}^{\doteq}$ if and only if its dual $(A \vdash B)^d$ is derivable in $\boldsymbol{LJ}^{\doteq}$. Furthermore, the dual of the derivation of $A \vdash B$ in $\boldsymbol{DLJ}^{\doteq}$ is a derivation of the dual of $A \vdash B$ in $\boldsymbol{LJ}^{\doteq}$*

**Remark** Analogous to the difference between $\to'$ and $\to$ in the case of dual-intuitionistic logic, the choice of which classical $\doteq$ rules to incorporate into intuitionistic logic makes a difference. Above I have stipulated selection of the alternative classical rules for $\doteq$ given in Figure 4. The singular-in-the-consequent restricted version of the original classical rule for $\doteq L$ given in Figure 2 would be:

$$\frac{A, X \vdash B}{A \doteq B, X \vdash} \doteq' L$$

$\doteq' L$ can be derived from $\doteq L$ in $\mathbf{LJ}^{\doteq}$

$$\cfrac{\cfrac{\cfrac{A, X \vdash B}{A \doteq B, X \vdash B} \doteq L_1}{A \doteq B, A \doteq B, X \vdash} \doteq L_2}{A \doteq B, X \vdash} WL$$

but $\doteq L$ would not be derivable if $\mathbf{LJ}^{\doteq}$ were formulated using $\doteq' L$

**Theorem 4** *In the case of dual-intuitionistic logic, $A \to B \equiv \neg A \vee B$*

**Proof** The following derivations in **DLJ**

$$\cfrac{\cfrac{\cfrac{A \vdash A}{\vdash A, A \to B} \to R_1}{\neg A \vdash A \to B} \neg L \qquad \cfrac{B \vdash B}{B \vdash A \to B} \to R_2}{\neg A \vee B \vdash A \to B} \vee L \qquad \cfrac{\cfrac{\cfrac{\cfrac{\neg A \vdash \neg A}{\neg A \vdash \neg A \vee B} \vee R_1}{\vdash A, \neg A \vee B} \neg R \qquad \cfrac{B \vdash B}{B \vdash \neg A \vee B} \vee R_2}{A \to B \vdash \neg A \vee B, \neg A \vee B} \to L}{A \to B \vdash \neg A \vee B} WR$$

■

**Theorem 5** *In the case of intuitionistic logic, $A \doteq B \equiv A \wedge \neg B$*

**Proof** Implied by Theorem 4 and Theorem 3 ■

This dual-intuitionistic logic is defined and investigated by Urbas in [19]. Several standard important properties for $\mathbf{DLJ}/\mathbf{DLJ}^{\doteq}$ are given in this paper, following are 6 of them:

**Theorem 6** ***(Cut Elimination)*** *The cut rule is eliminable from $\boldsymbol{DLJ}^{\doteq}$; every sequent which is derivable in this system has a cut-free derivation. ([19], p. 447.)*

**Theorem 7** *Every derivable sequent of the sentential fragment of $\boldsymbol{DLJ}^{\doteq}$ has a cut-free derivation with the Subformula Property. ([19], p. 448.)*

**Theorem 8** *Sentential $\boldsymbol{DLJ}^{\doteq}$ is decidable. ([19], p. 448.)*

**Theorem 9** ***DLJ***$^{\dot{=}}$ *is a conservative extension of* ***DLJ****. ([19], p. 448.)*

**Theorem 10** *There is no connective $\oplus$ definable in* ***DLJ*** *or* ***DLJ***$^{\dot{=}}$ *such that $A \vdash B$ is a derivable sequent if and only if $A \oplus B$ is a theorem. ([19], p. 450.)*

**Theorem 11** *$A \vdash B$ is a derivable sequent in* ***DLJ***$^{\dot{=}}$ *if and only if $A \doteq B$ is a counter-theorem; that is, the sequent $A \doteq B \vdash$ is derivable. Generally, $A \vdash B, Y$ is derivable in* ***DLJ***$^{\dot{=}}$ *if and only if $A \doteq B \vdash Y$ is also derivable. ([19], p. 450.)*

**Theorem 12** ***DLJ*** *has the same sentential theorems as* ***LK*** *([19], p. 443.)*

**Remark** Urbas labels Theorem 12 *dual-Glivenko*; which it is. More generally though, **DLJ**$^{\dot{=}}$ has the same sentential theorems as **LK**$^{\dot{=}}$.

I use this point to re-state the Glivenko theorem.

**Theorem 13** ***(dual-Glivenko)*** $A \vdash_{LK^{\dot{=}}}$ *iff* $\neg\neg A \vdash_{DLJ^{\dot{=}}}$

**Proof** If $A$ is a counter-theorem of **LK**$^{\dot{=}}$, then $\neg A$ is a theorem of **LK**$^{\dot{=}}$. Since $\neg A$ is a theorem of **LK**$^{\dot{=}}$, it is a theorem of **DLJ**$^{\dot{=}}$. Since $\neg A$ is a theorem of **DLJ**$^{\dot{=}}$, by $\neg L$, $\neg\neg A$ is a counter theorem of **DLJ**$^{\dot{=}}$. More generally, $A \vdash_{LK^{\dot{=}}} Y$ iff $\neg\neg A \vdash_{DLJ^{\dot{=}}} \neg\neg Y$, where $\neg\neg Y = \{\neg\neg y : y \in Y\}$ ■

**Theorem 14** *We can add the rule of double negation introduction (DNI) to* ***DLJ***$^{\dot{=}}$ *get classical logic*

$$\frac{\neg\neg A \vdash Y}{A \vdash Y}\ DNI$$

**Proof** Firstly, with the DNI rule the sequent $A \vdash \neg\neg A$ is derivable

$$\frac{\neg\neg A \vdash \neg\neg A}{A \vdash \neg\neg A}\ DNI$$

Now, if the sequent $x_1, ..., x_n \vdash Y$ is derivable in **LK** then the sequent $\vdash Y, \neg x_1, ..., \neg x_n$ is derivable in **DLJ**+DNI. To see this, consider the sequents:

1. $x_n \vdash Y, \neg x_n, ..., \neg x_{n-1}$
2. $\vdash Y, \neg x_n, ..., \neg x_{n-1}, \neg x_n$

(2) is derivable from (1), through $\neg R$. (1) is also derivable from (2): first derive $\neg\neg x_n \vdash Y, \neg x_n, ..., \neg x_{n-1}$ from (2) using $\neg L$. Then use cut, with $x_n \vdash \neg\neg x_n$, to get (1).

Given this, formulas on the left can be freely moved to the right and back. As a consequence, any classical sequent $x_1, ..., x_n \vdash Y$ can be represented by a dual-intuitionistic sequent $\vdash Y, \neg x_1, ..., \neg x_n$. If one wants to apply a rule of classical sequent calculus that needs formulas on the left, one can move the necessary formula to the left, apply the corresponding rule in dual-intuitionistic sequent calculus, and move the resulting formula back to the right. ■

Here are some other results for dual-intuitionistic logic.

**Theorem 15** *If $A$ is a propositional formula containing only $\vee$ and $\neg$ then $A \vdash_{LK}$ iff $A \vdash_{LDJ}$.*

**Proof** That $A \vdash_{LDJ}$ implies $A \vdash_{LK}$ is obvious. For the other direction, consider $A$ as a disjunction of $n$ formulas each of which is not a disjunction and is therefore either a proposition letter or begins with $\neg$. Since $A$ is a counter-theorem, each of these $n$ sub-formulas is a counter-theorem of classical logic. Since no atomic proposition is a counter-theorem in classical logic, each sub-formula is a negation and by the dual-Glivenko result is also a counter-theorem of dual-intuitionistic logic. ■

**Theorem 16** ***(Interpolation Theorem)*** *If $A \vdash Y$ then there exists an interpolant for the pair $\langle A, Y \rangle$; that is, a formula $F$ such that $A \vdash F$ and $F \vdash Y$, and every schematic letter occurring in $F$ occurs in both $A$ and in some formula of $Y$.*

**Proof** The interpolation theorem for dual-intuitionistic logic is implied by the interpolation theorem for intuitionistic logic and the duality correspondence between these two logics as given in Theorem 3. ■

Intuitionistic logic has the following two properties, known as the *disjunction property* and *existence property* respectively:

**Theorem 17** $\vdash A \vee B$ *iff* $\vdash A$ *or* $\vdash B$.

**Theorem 18** $\vdash (\exists x)A(x)$ *iff* $\vdash A[x/t]$ *for some term* $t$.

Dual-intuitionistic logic has two corresponding properties, which I shall term the *conjunction property* and *universality property* respectively:

**Theorem 19** *(Conjunction Property)* $A \wedge B \vdash$ *iff* $A \vdash$ *or* $B \vdash$

**Proof** Let $A \wedge B \vdash$ be the conclusion of a derivation in $\mathbf{DLJ}^{\dot{-}}$. Then it can only have been derived by a $\wedge L$ rule, either from $A \vdash$ or from $B \vdash$. The converse is obvious. ■

**Theorem 20** *(Universality Property)* $(\forall x)A(x) \vdash$ *iff* $A[x/t] \vdash$ *for some term* $t$.

**Proof** Let $(\forall x)A(x) \vdash$ be the conclusion of a derivation in $\mathbf{DLJ}^{\dot{-}}$. Then it can only have been derived by the $\forall L$ rule from $A[x/t] \vdash$ for some term $t$. The converse is obvious. ■

**Definition 2** *The Harrop formulas are inductively defined by:*

*(1) Every atomic formula is a Harrop formula,*

*(2) If $B$ is a Harrop formula and $A$ is an arbitrary formula, then $A \rightarrow B$ is a Harrop formula*

*(3) If $B$ is a Harrop formula, then $(\forall x)B$ is a Harrop formula.*

*(4) If $A$ and $B$ are Harrop formulas, then so is $A \wedge B$*

*(5) If $A$ is an arbitrary formula, then $\neg A$ is a Harrop formula.*

**Theorem 21** *(Harrop, 1960) Let $X$ be a cedent in* **LJ** *containing Harrop formulas, and let $A$ and $B$ be arbitrary formulas*

*(a) If $X \vdash_{LJ} (\exists x)B(x)$, then there exists a term $t$ such that $X \vdash_{LJ} B[x/t]$*

*(b) If $X \vdash_{LJ} A \vee B$, then at least one of $X \vdash_{LJ} A$ and $X \vdash_{LJ} B$*

I will now show the corresponding theorem for dual-intuitionistic logic.

**Definition 3** *The dual-Harrop formulas are inductively defined by:*

*(1) Every atomic formula is a dual-Harrop formula,*

*(2) If $A$ is a dual-Harrop formula and $B$ is an arbitrary formula, then $A \doteq B$ is a dual-Harrop formula*

*(3) If $B$ is a dual-Harrop formula, then $(\exists x)B$ is a dual-Harrop formula.*

*(4) If $A$ and $B$ are dual-Harrop formulas, then so is $A \vee B$*

*(5) If $A$ is an arbitrary formula, then $\neg A$ is a dual-Harrop formula.*

**Theorem 22** *Let $Y$ and $Y'$ be cedents containing dual-Harrop formulas, and let $A$, $B$, $C$ and $D$ be arbitrary formulas*

*(a) If $A \wedge B \vdash_{DLJ^{\doteq}} Y$, then at least one of $A \vdash_{DLJ^{\doteq}} Y$ and $B \vdash_{DLJ^{\doteq}} Y$*

*(b) If $(\forall x)B \vdash_{DLJ^{\doteq}} Y$, then there exists a term $t$ such that $\boldsymbol{DLJ^{\doteq}}$ proves $B[x/t] \vdash_{DLJ^{\doteq}} Y$*

**Proof** We use induction on the length of cut-free $\mathbf{DLJ}^{\doteq}$-proofs. Firstly, a proof of part (a). There are a number of ways we can get a sequent of the form $A \wedge B \vdash Y$ in accordance with the theorem.

1. An application of a $\wedge L$ rule to get a sequent of the form $A \wedge B \vdash Y$. This sequent could only be derived using these rules if $A \vdash Y$ or $B \vdash Y$.

2. Application of a $\vee R$ rule to get a sequent of the form $A \wedge B \vdash C \vee D, Y$. This sequent could only be derived using these $\vee$ rules if $A \wedge B \vdash C, Y$ or $A \wedge B \vdash D, Y$:

   - in the case of $A \wedge B \vdash C, Y$, by 1 either (i) $A \vdash C, Y$ or (ii) $B \vdash C, Y$. In the case of (i), it follows that $A \vdash C \vee D, Y$ by $\vee R_1$. In the case of (ii), it follows that $B \vdash C \vee D, Y$ by $\vee R_1$
   - in the case of $A \wedge B \vdash D, Y$, by 1 either (i) $A \vdash D, Y$ or (ii) $B \vdash D, Y$. In the case of (i), it follows that $A \vdash C \vee D, Y$ by $\vee R_2$. In the case of (ii), it follows that $B \vdash C \vee D, Y$ by $\vee R_2$

3. Application of the $\doteq R$ rule to get a sequent as follows:

$$\frac{A \wedge B \vdash Y, C \qquad D \vdash Y'}{A \wedge B \vdash C \doteq D, Y, Y'} \doteq R$$

   Here we are interested in the top-left premise $A \wedge B \vdash Y, C$. Since $C$ is a dual-Harrop formula by definition, $A \vdash C, Y$ or $B \vdash C, Y$. If $A \vdash C, Y$, then

$$\frac{A \vdash C, Y \qquad D \vdash Y'}{A \vdash C \doteq D, Y, Y'} \doteq R$$

   if $B \vdash C, Y$ then

$$\frac{B \vdash C, Y \qquad D \vdash Y'}{B \vdash C \doteq D, Y, Y'} \doteq R$$

4. Application of the negation rules can not introduce a sequent of the form $A \wedge B \vdash Y$. The structural rules $WR$ and $KR$ can not introduce a conjunction on the left hand side. The structural rule $KL$ can, although since a derivation of the sequent $A \wedge B \vdash Y$ using the $KL$ rule is preceded by $\vdash Y$, it trivially follows that $A \vdash Y$ or $B \vdash Y$ using $KL$ itself.

Secondly, part (b). There are three ways to derive a sequent of the form $(\forall x)B \vdash Y$: $\forall L$, $\exists R$ and $\forall R$. Out of these, only the first two can have $Y$ as a set dual-Harrop formula, so attention can be confined to these two.

1. In the case of the $\forall L$ rule, $(\forall x)B \vdash Y$ is explicitly derived from $B[x/t] \vdash Y$ for some term $t$.

2. In the case of $\exists R$, $(\forall x)B \vdash (\exists x)F, Y$ is derived from $(\forall x)B \vdash F[x/c], Y$, which is derived from $B[x/t] \vdash F[x/c], Y$. By $\exists R$, $B[x/t] \vdash (\exists x)F, Y$

■

Just as intuitionistic negation can be defined as $A \to \bot$, dual-intuitionistic negation can be defined as $\top \doteq A$, with the following rules for the *verum* constant $\top$:

$$\top \text{ rules} \quad \frac{\vdash Y}{\top \vdash Y} \qquad \vdash \top$$

**Theorem 23** *$\neg A$ can be defined as $\top \doteq A$ in* $\boldsymbol{DLJ}^{\doteq}$

**Proof** The following derivations

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \neg A}\neg R}{\top \vdash A, \neg A}}{\top \doteq A \vdash \neg A} \doteq L \qquad \dfrac{\dfrac{\vdash \top \qquad A \vdash A}{\vdash \top \doteq A, A} \doteq R}{\neg A \vdash \top \doteq A} \neg L$$

■

**Lemma 1** *Let $A \leftrightarrow B$ be $(A \to B) \wedge (B \to A)$. Let $D_n$ be the disjunction of all sentences of the form $p_i \leftrightarrow p_j$, for $0 \leq i \leq j \leq n$. For no $n$ is $D_n$ a theorem of intuitionistic logic.*

**Definition 4** *Let $A \div B$ be $(A \doteq B) \vee (B \doteq A)$. Let $C_n$ be the conjunction of all sentences of the form $p_i \div p_j$, for $0 \leq i \leq j \leq n$.*

**Lemma 2** *For no $n$ is it the case that $C_n$ is a counter-theorem of* $\boldsymbol{DLJ}^{\doteq}$.

**Proof** Suppose that there was an $n$ such that $C_n$. Going by the duality mapping, it follows that there is an $n$ such that $D_n$ is a theorem of intuitionistic logic. But according to Lemma 1, there is no such $D_n$.

**Theorem 24** *There is no finitely many-valued logic which characterises* $\boldsymbol{DLJ}^{\doteq}$

**Proof** The proof is by contradiction. Suppose that there were, and that it had $n$ truth values. Let $\mathcal{V}$ stand for the set of truth values in this logic and $\mathcal{D} \subseteq \mathcal{V}$ stand for the set of designated truth values in this logic.

Since $A \wedge B \vdash_{DLJ^{\doteq}} A$:

- if $A \notin \mathcal{D}$ then $A \wedge B \notin \mathcal{D}$

Since $A \vdash_{DLJ^{\doteq}} A \vee B$ and $B \vdash_{DLJ^{\doteq}} A \vee B$:

- if $A \notin \mathcal{D}$ and $B \notin \mathcal{D}$ then $A \vee B \notin \mathcal{D}$

Since $p \doteq p \vdash_{DLJ^{\doteq}}$

- for all $v \in \mathcal{V}$, $f_{\doteq}(v, v) \notin \mathcal{D}$

Now, consider any interpretation, $v$. Since there are only $n$ truth values, for some $i$, $j$ such that $0 \leq i \leq j \leq n, v(p_i) = v(p_j)$. Hence, by (iii):

- $v(p_i \doteq p_j) \notin \mathcal{D}$ and $v(p_j \doteq p_i) \notin \mathcal{D}$.

By (ii):

- $v(p_i \div p_j) \notin \mathcal{D}$.

By (i):

- $v(C_n) \notin \mathcal{D}$.

Thus, $C_n$ is a counter-theorem, but according to Lemma 2, it can't be. ■

It is by now apparent that dual-intuitionistic logic is a paraconsistent logic. Following is a survey of some derivable and non-derivable sequents in $\mathbf{DJL}^{\dot{-}}$ accompanied by a comparative survey of how these sequents fare in the established paraconsistent logic, *Logic of Paradox* (**LP**).[3] This exercise is intended to help form a picture of the nature of dual-intuitionistic logic, particularly as a paraconsistent logic, and will serve as a reference point for following discussion.

| | **DLJ** | **LP** | | **DLJ** | **LP** |
|---|---|---|---|---|---|
| $\neg\neg A \vdash A$ | ✓ | ✓ | $A \vdash \neg\neg A$ | × | ✓ |
| $\vdash A \vee \neg A$ | ✓ | ✓ | $A \wedge \neg A \vdash$ | × | × |
| $\neg A \vee \neg B \vdash \neg(A \wedge B)$ | ✓ | ✓ | $\neg(A \wedge B) \vdash \neg A \vee \neg B$ | ✓ | ✓ |
| $\neg(A \vee B) \vdash \neg A \wedge \neg B$ | ✓ | ✓ | $\neg A \wedge \neg B \vdash \neg(A \vee B)$ | × | ✓ |
| $A \vdash B \to A$ | ✓ | ✓ | $A \vdash \neg A \to B$ | × | × |
| $\neg A \to \neg B \vdash B \to A$ | ✓ | ✓ | $A \to B \vdash \neg B \to \neg A$ | × | ✓ |
| $\neg A \vdash A \to B$ | ✓ | ✓ | $\neg A \to B \vdash \neg B \to A$ | × | ✓ |
| $(A \to B) \wedge \neg B \vdash \neg A$ | × | × | $A \wedge (A \to B) \vdash B$ | × | × |
| $(A \to B) \wedge (B \to C) \vdash A \to C$ | × | × | $\vdash \neg(A \wedge \neg A)$ | ✓ | ✓ |
| $A \dot{-} A \vdash$ | ✓ | × | $\neg A \dot{-} B \vdash \neg B \dot{-} A$ | ✓ | ✓ |
| $A \dot{-} B \vdash A \wedge \neg B$ | ✓ | ✓ | $A \dot{-} A \vdash B$ | ✓ | × |
| $\neg A \dot{-} \neg B \vdash B \dot{-} A$ | ✓ | ✓ | $A \vdash B, A \dot{-} B$ | ✓ | ✓ |
| $A \dot{-} \neg\neg A \vdash$ | × | × | $A \dot{-} B \vdash \neg B \dot{-} \neg A$ | × | ✓ |
| $A \wedge \neg B \vdash A \dot{-} B$ | × | ✓ | $A \wedge (B \dot{-} A) \vdash$ | × | × |
| $A \dot{-} \neg B \vdash B \dot{-} \neg A$ | × | ✓ | $(A \wedge \neg A) \dot{-} B \vdash$ | × | × |
| $(\forall x)F(x) \vdash (\exists x)F(x)$ | ✓ | ✓ | $\neg(\forall x)F(x) \vdash (\exists x)\neg F(x)$ | ✓ | × |
| $\neg(\forall x)\neg F(x) \vdash (\exists x)\neg F(x)$ | ✓ | ✓ | $\neg(\exists x)F(x) \vdash (\forall x)\neg F(x)$ | ✓ | × |
| $\neg(\exists x)\neg F(x) \vdash (\forall x)F(x)$ | ✓ | ✓ | $\vdash (\exists x)F(x) \vee (\exists x)\neg F(x)$ | ✓ | × |
| $(\forall x)F(x) \vdash \neg(\exists x)\neg F(x)$ | × | ✓ | $(\forall x)\neg F(x) \vdash \neg(\exists x)F(x)$ | × | ✓ |
| $(\exists x)F(x) \vdash \neg(\forall x)\neg F(x)$ | × | ✓ | $\neg(\exists x)F(x) \vdash (\forall x)\neg F(x)$ | × | ✓ |
| $(\exists x)(F(x) \wedge \neg F(x)) \vdash$ | × | × | $\vdash \neg(\exists x)(F(x) \wedge \neg F(x))$ | × | ✓ |

Figure 11: Derivable and non-derivable sequents

**Theorem 25** *The dual-Glivenko theorem does not extend to predicate logic*

**Proof** The last sequent in the above list $\vdash \neg(\exists x)(F(x) \wedge \neg F(x))$ is classically derivable

## 2 Kripke Semantics

With the knowledge that intuitionistic logic enjoys a natural and informative Kripke semantics, we here take a look at Kripke semantics for dual-intuitionistic logic. There are essentially two approaches which can be taken. The first one, which I have gathered from [7] and [15] is based upon the familiar Kripke semantics for intuitionistic logic.[4] An interpretation for intuitionistic logic is a structure $\langle W, R, v\rangle$, where $W$ is the set of worlds, $R$ is the accesibility relation and $v$ is the valuation function. $R$ is characterised by that of the normal modal logic **S4**, so it is reflexive and transitive. There is one further constraint, which is that for every atomic proposition, $p$:

$$\text{for all } w \in W, \text{ if } v_w(p) = 1 \text{ and } wRw', v_{w'}(p) = 1$$

This condition is termed the *heredity condition.* Coupling this interpretation definition with the following defined connectives essentially give us the first approach to a Kripke semantics for dual-intuitionistic logic:

[3] Appendix A contains a sequent formalisation of LP. $A \dot{-} B := A \wedge \neg B$

[4] this approach can be associated with the addition of dual-intuitionistic $\dot{-}$ and $\neg$ to intuitionistic logic

$v_w(\top) = 1$ for every $w \in W$

$v_w(A \wedge B) = 1$ iff $v_w(A) = 1$ and $v_w(B) = 1$

$v_w(A \vee B) = 1$ iff $v_w(A) = 1$ or $v_w(B) = 1$

$v_w(A \doteq B) = 1$ iff there is some $w'$ such that $w'Rw$, $v_{w'}(A) = 1$ and $v_{w'}(B) = 0$

$v_w(\neg A) = 1$ iff there is some $w'$ such that $w'Rw$ and $v_{w'}(A) = 0$

Recall that $\neg A$ can be defined as $\top \doteq A$ and $A \to B$ can be defined as $\neg A \vee B$.

The second approach, which I now present, better captures the notion of constructive falsity associated with dual-intuitionistic logic. As above, an interpretation is a structure $\langle W, R, v\rangle$. $R$ is symmetric and transitive. The heredity condition is replaced by the following condition:

$$\text{for all } w \in W, \text{ if } v_w(p) = 0 \text{ and } wRw', v_{w'}(p) = 0$$

Call this the *dual-heredity condition.* The connectives are defined as follows:

$v_w(A \wedge B) = 1$ iff $v_w(A) = 1$ and $v_w(B) = 1$

$v_w(A \vee B) = 1$ iff $v_w(A) = 1$ or $v_w(B) = 1$

$v_w(\neg A) = 1$ iff there is some $w'$ such that $wRw', v_{w'}(A) = 0$

$v_w(A \to B) = 1$ iff there is some $w'$ such that $wRw', v_{w'}(A) = 0$ or $v_{w'}(B) = 1$

$v_w(A \doteq B) = 1$ iff there is some $w'$ such that $wRw', v_{w'}(A) = 1$ and $v_{w'}(B) = 0$

Tableaux rules are given in Figure 12

I shall refer to this system as **DI**. As with Kripke semantics for intuitionistic logic, these Kripke semantics for dual-intuitionistic help elucidate some of the notions it captures. They reflect the notion that our current knowledge about the falsity of statements can increase. Some statements whose falsity status was previously indeterminate can down the track be established as false. The value *false* corresponds to firmly established falsity that is preserved with the advancement of knowledge whilst the value *true* corresponds to "not false yet".

Furthermore, as with Kripke semantics for intuitionistic logic, it gives us a straightforward way to devise logics intermediate between dual-intuitionistic logic and classical logic, by strengthening the frame conditions. For example, with $R$ characterised by the convergent frames of the modal logic **S4.2** (**S4** $+\Diamond\Box \to \Box\Diamond$), we can prove $\neg A \wedge \neg\neg A \vdash$, which is not provable in **DI**. With $R$ characterised by the connected frames of the logic **S4.3** (**S4** $+\Box(\Box p \to q) \vee \Box(\Box q \to p)$, we can prove $A \doteq B \wedge B \doteq A \vdash$, which is not provable in **DI**.[5]

Tableaux demonstrations of these facts are given in Figure 13

[5]These 'super-dual-intuitionistic' logics would be the counterparts to *Logic of Weak Excluded Middle* and *Godel-Dummett* logic respectively

$$\begin{array}{c} A \wedge B, +i \\ | \\ A, +i \\ B, +i \end{array} \qquad \begin{array}{c} A \wedge B, -i \\ \diagup \quad \diagdown \\ A, -i \quad B, -i \end{array}$$

$$\begin{array}{c} A \vee B, +i \\ \diagup \quad \diagdown \\ A, +i \quad B, +i \end{array} \qquad \begin{array}{c} A \vee B, -i \\ | \\ A, -i \\ B, -i \end{array}$$

$$\begin{array}{c} A \rightarrow B, +i \\ | \\ iRj \\ \diagup \quad \diagdown \\ A, -j \quad B, +j \end{array} \qquad \begin{array}{c} A \rightarrow B, -i \\ iRj \\ | \\ A, +j \\ B, -j \end{array}$$

$$\begin{array}{c} A \doteq B, +i \\ | \\ iRj \\ A, +j \\ B, -j \end{array} \qquad \begin{array}{c} A \doteq B, -i \\ iRj \\ \diagup \quad \diagdown \\ A, -j \quad B, +j \end{array}$$

$$\begin{array}{c} \neg A, +i \\ | \\ iRj \\ A, -j \end{array} \qquad \begin{array}{c} \neg A, -i \\ iRj \\ | \\ A, +j \end{array}$$

| dual-heredity | reflexivity | transitivity |
|---|---|---|
| $\begin{array}{c} p, -i \\ iRj \\ \vert \\ p, -j \end{array}$ | $\begin{array}{c} \vert \\ iRi \end{array}$ | $\begin{array}{c} iRj \\ jRk \\ \vert \\ iRk \end{array}$ |

Figure 12: Tableaux rules for dual-intuitionistic logic

**S4.2** frame condition,
where $l$ is new to the branch

$$\begin{array}{c} iRj \\ iRk \\ | \\ jRl \\ kRl \end{array}$$

**S4.3** frame condition

$$\begin{array}{cc} \multicolumn{2}{c}{iRj} \\ \multicolumn{2}{c}{iRk} \\ jRk & kRj \end{array}$$

$$\neg A \wedge \neg\neg A \vdash$$

$$\begin{array}{c} \neg A \wedge \neg\neg A, +0 \\ | \\ \neg A, +0 \\ \neg\neg A, +0 \\ | \\ 0R1 \\ A, -1 \\ | \\ 0R2 \\ \neg A, -2 \\ | \\ 1R3 \\ 2R3 \\ | \\ A, +3 \\ | \\ A, -3 \\ | \\ \times \end{array}$$

$$A \doteq B \wedge B \doteq A \vdash$$

$$\begin{array}{cc} \multicolumn{2}{c}{A \doteq B \wedge B \doteq A, +0} \\ \multicolumn{2}{c}{|} \\ \multicolumn{2}{c}{A \doteq B, +0} \\ \multicolumn{2}{c}{B \doteq A, +0} \\ \multicolumn{2}{c}{|} \\ \multicolumn{2}{c}{0R1} \\ \multicolumn{2}{c}{A, +1} \\ \multicolumn{2}{c}{B, -1} \\ \multicolumn{2}{c}{|} \\ \multicolumn{2}{c}{0R2} \\ \multicolumn{2}{c}{B, +2} \\ \multicolumn{2}{c}{A, -2} \\ 1R2 & 2R1 \\ B, -2 & A, -1 \\ | & | \\ \times & \times \end{array}$$

Figure 13: ‘Super-dual-intuitionistic’ tableaux demonstrations

With $R$ characterised by that of **S5**, both $A \wedge \neg A \vdash$ and $A \vdash \neg\neg A$ and we get classical logic. We can see that these two derivations stand and fall together:

$$
\begin{array}{ccc}
\begin{array}{c}
A \wedge \neg A, +0 \\
| \\
A, +0 \\
\neg A, +0 \\
0R0 \\
| \\
0R1 \\
1R1 \\
A, -1
\end{array}
& \qquad &
\begin{array}{c}
A, +0 \\
\neg\neg A, -0 \\
0\mathrm{R}0 \\
| \\
\neg A, +0 \\
| \\
0R1 \\
1R1 \\
A, -1 \\
| \\
\neg A, +1 \\
| \\
1R2 \\
2R2 \\
0R2 \\
A, -2 \\
| \\
\vdots
\end{array}
\end{array}
$$

For either of these two tableaux to close, the symmetry condition is required.

Heading in the other direction, one could weaken the conditions on the Kripke semantics to devise 'sub-dual-intuitionistic' logics.[6]

These Kripke semantics could be straightforwardly extended to accomodate quantification. I cursorily suggest that the resulting quantified modal logic would be variable domain and conditions would be that for all $w \in W$, if $wRw'$ then:

- $v_w P(a_1, ..., a_n) = 0$ then $v_{w'} P(a_1, ..., a_n) = 0$
- $D_w \subseteq D_{w'}$

where $D_w$ stands for the domain of quantification at world $w$. The truth conditions for the quantifiers would be:

- $v_w(\exists x A) = 1$ iff for some $w'$ such that $wRw'$ and some $d \in D_{w'}$, $v_w(A[x/d]) = 1$
- $v_w(\forall x A) = 1$ iff for all $d \in D_w$, $v_w(A[x/d]) = 1$

## 3 A Logic for Scientific Method

* Given a number of characteristics of dual-intuitionistic logic, it particularly strikes me as being a logic potentially suitable for formalising aspects of scientific method. This is not the occasion, nor am I sufficiently abreast with developments in the philosophy of science and scientific practice, to offer a complete, unified account; I can only hope to make a few separate tentative observations.

I would like to begin by making some preliminary points with regard to interpreting dual-intuitionistic logic.

[6] this idea with regard to intuitionistic logic is investigated in [14]

- Restall [16] gives a viable framework for analysing logical consequence in terms of the concepts of assertion (acceptance) and denial (rejection). This account uses the notion of a *state*. The notation $[X : Y]$ represents states, where $X$ and $Y$ are sets of statements. Restall suggests that "a state might be used to represent the *outlook* of an agent which we take to *accept* each statement in $X$ and *reject* each statement in $Y$. We might also use a state to represent the *context* in some dialogue or discourse at which each statement $X$ is *asserted* and each statement in $Y$ in *denied*" ([16], p. 7.). The state $[X : Y]$ corresponds to the sequent/argument $X \vdash Y$. Basically, if $X \vdash Y$, then the state $[X : Y]$ is *incoherent*; it is incoherent to assert all of $X$ and deny all of $Y$. Conversely, if $X \nvdash Y$ then the state $[X : Y]$ is *coherent*.

  This account of logical consequence provides a neutral, common vocabulary with which classical and non-classical logics can be understood. The assertion of $\neg A$ need not have the same significance as the assertion of $A$ under this account. For the paraconsistentist, sometimes it is appropriate to assert both $A$ and $\neg A$, so the state $[A \wedge \neg A :]$ need not be incoherent. In terms of this vocabulary, whilst the rejection of the $\neg L$ inference is a customary way to achieve paraconsistency, we have seen in this paper that a challenge can be mounted against the logical principle *ex contradictione quodlibet* by imposing the restriction that no more than one statement can be asserted per any given state.

- A prime motivation for Restall's account is the need for a resource to analyse logical consequence as relating arguments with multiple premises and multiple conclusions. With such a resource, Gentzen's multiple conclusion sequent calculus can be accommodated, contra the skeptics.

  However, issues concerning the jump from intuitionistic sequents to classical sequents I here put aside, as I am concerned with explicating a way to read dual-intuitionistic sequents.

  With intuitionistic logic, when an argument $x_1, x_2, ..., x_n \vdash A$ is valid, put crudely, we may say that if $x_1$ holds and $x_2$ holds ... and $x_n$ holds, then $A$ holds. The key points here are that we read sequents from left-to-right and go from the assertion of multiple statements to the assertion of *no more than one* conclusion.

  Taking this approach to dual-intuitionistic logic, when an argument $A \vdash y_1, y_2, ..., y_n$ is valid, put crudely, we may say that if $A$ holds, then $y_1$ holds or $y_2$ holds ... or $y_n$ holds. In this light, dual-intuitionistic logic is a multiple conclusion calculus. Alternatively though, we may read dual-intuitionistic sequents from right-to-left. In this case, with an argument $A \vdash y_1, y_2, ..., y_n$, put crudely, we may say that if $y_1$ does not hold and $y_2$ does not hold ... and $y_n$ does not hold, then $A$ does not hold; if $A$ does hold, then something is wrong. We go from the denial of multiple statements to the denial of *no more than one* conclusion. This reversed perspective seems less natural, though is nonetheless comprehensible and quite conceivably better suited to certain practices.

- I identify an abstraction/template of the framework outlined in [16] with the following ideas:

  1. Objects of the form $[X : Y]$, which can be termed 'states', can be correlated with logical sequents/arguments $X \vdash Y$.
  2. With the object $[X : Y]$, $X$ and $Y$ are sets of things. Take $\alpha$ and $\beta$ to be two terms, which are in some required sense antithetical. In Restall's case, $\alpha$ is 'accepted' and $\beta$ is 'rejected'. Each of the things in $X$ are said to be $\alpha$ and each of the things in $Y$ are said to be $\beta$.
  3. Take $\gamma$ to be a positive term and $\delta$ to be a negative term, which is the opposite of $\gamma$. When $X \vdash Y$, $[X : Y]$ is $\delta$. When $X \nvdash Y$, $[X : Y]$ is $\gamma$. In Restall's case, $\gamma$ is coherent and $\delta$ is incoherent.

This template can be adopted and filled with terms as dictated by context. For example, with regard to the potential applicability of dual-intuitionistic logic to formalising scientific method, here are some cursory examples. $[X : Y]$ can be termed 'states', 'theories', 'experiments'. $\alpha$ can be termed 'asserted', 'postulated', 'observed'. $\beta$ can be termed 'denied', 'disputed', 'unobserved'. $\gamma$ can be termed 'coherent', 'acceptable'. $\delta$ can be termed 'incoherent', 'unacceptable'.

- Work on developing an informal interpretation for dual-intuitionistic logic is something which remains to be sufficiently addressed; in the way that, for example, the informal Brouwer-Heyting-Kolmogorov explication of intuitionistic truth naturally motivates formalized intuitionistic logic. Here is a tentative interpretation
  - A falsification of $A \wedge B$ consists of a falsification of $A$ or a falsification of $B$
  - A falsification of $A \vee B$ consists of a falsification of $A$ and a falsification of $B$
  - A falsification of $\neg A$ consists of a confirmation of $A$
  - A falsification of $A \to B$ consists of a confirmation of $A$ and a falsification of $B$
  - A falsification of $A \doteq B$ consists of a falsification of $A$ or confirmation of $B$

* The presence of inconsistency in science is something which is being increasingly recognised. Priest [12] notes three different types of inconsistency in science: between theory and observation, between theory and theory, and internal to a theory.

With inconsistency between theory and observation, there is a well-received theory $T$, with a certain observable consequence, $o$, but after conducting an experiment, $\neg o$ is observed. Rather than rejecting $T$ or rejecting $\neg o$, the contradiction may be considered anomalous and both $T$ and $\neg o$ may be provisionally accepted. $T$ will not be abandoned since there is no better alternative and $\neg o$ will not be abandoned until we have a reason why the observation was incorrect.

One example of this type of inconsistency: in the first half of the 19th century, astronomers were observing the path of the planet Uranus to see if it conformed to the path predicted by Newton's theory of gravitation. It didn't. Yet both the theory, with its observational consequence and the contradictory observation were maintained, until an explanation became available; in this case, that an unknown planet, namely Neptune, was affecting the path of Uranus

The second kind of inconsistency, between theory and theory, is when there are two well-accepted theories, $T_1$ and $T_2$, which have inconsistent consequences. Despite the fact that this may be problematic, both $T_1$ and $T_2$ are retained until a suitable replacement theory for one or both is found.

The third example of an inconsistency is when a theory is self-inconsistent, in particular, where the inconsistencies are located internally, away from observational consequences. An often cited example is Bohr's theory of the atom, which included both mutually inconsistent classical electrodynamic principles and quantum principle. Since the empirical predications of this theory were at the time much better than alternative theories, this problematic aspect of the theory was tolerated.

So as I have summarily explained, it is quite possible that inconsistency may be present in scientific investigation and systems. Yet scientists infer things in a methodical and rational fashion; they do not infer arbitrary conclusions, not anything goes. So the inference procedure and general logical framework employed in such contexts should be a paraconsistent one, where an arbitrary $A$ and $\neg A$ do not entail an arbitrary $B$.

Thus in its capacity as a logic of scientific method, the paraconsistency of dual-intuitionistic logic is of value. How exactly to interpret sequents and invoke the state ($[X : Y]$) template resource described above I leave untended, but note that this will vary according to context and which type of the three inconsistencies are being dealt with.

Furthermore, listed in Figure 11 are a collection of idiosyncratically underivable sequents in dual-intuitionistic logic. For example, according to dual-intuitionistic logic, the states $[A : \neg\neg A]$ and $[\neg A \wedge \neg B : \neg(A \vee B)]$ need not be incoherent. What can be made of these facts remains to be addressed.

* Karl Popper's *falsificationism* reverses the view that accumulated experience and data leads to scientific generalisations and hypotheses. On the contrary, the scientific method is the formation of bold hypotheses that are then subject to rigorous testing. The hypotheses that survive the testing process

constitute current scientific knowledge.[9] So for Popper, the notion of falsity (falsification, refutation) is more important in scientific research than the notion of truth (verification, proof).[17]

Insofar as dual-intuitionistic logic is a logic of constructive falsity, it can be associated with falsificationism. In light of the Kripke semantics given, we can think of a world as a state of scientific information at a certain time. The things that do not hold at it are those things which are falsified at this time. When one state $w$ accesses another state $w'$, $w'$ is considered an extension of $w$, obtained by accumulating, possibly zero, falsifications. Furthermore, if something is falsified, it stays falsified. Thus, the real test of a statement consists in its falsification, not in its verification. The set of refuted statements is increasing whereas the set of acceptable statements (conjectures) is decreasing.

However, there are points of contrast between the principles of falsificationism and dual-intuitionistic logic. Firstly, as Priest ([12], p. 122.) points out, given a well-received theory $T$ with a certain observable consequence $o$, and the observation of $\neg o$ after conducting an experiment, the simple minded falsificationist would suggest that this shows that $T$ is wrong and is to be discarded. Popper himself states that the acceptance of inconsistency "... would mean the complete breakdown of science" since an inconsistent system is ultimately uninformative.[7] As supported and to the contrary, dual-intuitionistic logic encourages the cautious adoption of inconsistency tolerance.

The inference of modus tollens: $A \to B, \neg B \vdash \neg A$, is the cornerstone of falsificationism. As Popper puts it in the Logic of Scientific Discovery:

> The falsifying mode of inference here referred to – the way in which the falsification of a conclusion entails the falsification of the system from which it is derived – is the modus tollens of classical logic.

The falsification of statements and theories occurs through *modus tollens*, via some observation. Following is a symbolisation of this process. Suppose some universal statement $U$ or theory $T$ implies an observation $o$:

$$U \to o \text{ or } T \to o$$

An observation conflicting with $o$, however is made:

$$\neg o$$

so by *modus tollens*

$$\neg U \text{ or } \neg T$$

A criticism of naive falsificationism, which Popper himself anticipated, is the thesis termed *confirmation holism* by Quine, also known as the Duhem-Quine thesis. In short, according to this thesis, a single scientific hypothesis cannot be tested in isolation, since other, auxiliary hypotheses will always be needed to draw empirical consequences from it. Consequently, a single hypothesis may be retained in the face of any adverse empirical evidence, if one is prepared to make modifications elsewhere in the system.

This brings me to a second point of contrast between falsificationism and dual-intuitionistic logic; the practice of *modus tollens* in the former and the fact that $(A \to B) \wedge \neg B \nvdash \neg A$ in the latter. Perhaps these issues and the failure of *modus tollens* in dual-intuitionistic logic should motivate the search for an alternative valid inferential practice. A tentative idea is to base inferences on the $\doteq$ operator.

Two pretty much equivalent suggestions for interpreting $A \doteq B$ are as "A excludes B" and "A without B". Although neither *modus ponens* nor *modus tollens* hold in dual-intuitionistic logic, their dual counterparts naturally do:

(a) $B \vdash A, B \doteq A$

(b) $\neg A \vdash \neg B, B \doteq A$

[7] [5]

Perhaps the validity of these inferences and the right-to-left reading of sequents outlined earlier can be put to good use. For example, (a) can be interpreted in a variety of ways:

- Assuming the falsity of $A$ and the falsity of $B \doteq A$, one can assume the falsity of $B$.
- Rejecting $A$ and rejecting $B \doteq A$ yet accepting $B$ is incoherent.
- There is a theory $T$. It is denied that this theory excludes an observation $o$; $T \doteq o$ is denied. So given $T$ there should be an accompanying observation $o$; $T$ includes the observation $o$. Yet $o$ is unobserved, therefore the falsity/falsification of $T$ can be inferred.

* Finally, a brief word on quantification in dual-intuitionistic logic. Earlier a list of derivable and non-derivable sequents involving quantification was given. The derivable sequents are also classically derivable. The non-derivable sequents are also classically derivable and their non-derivability in dual-intuitionistic logic, particularly those involving a relation between universal and existential quantification, is characteristic of the nature of quantification in this context. This particularity prompts a consideration of how dual-intuitionistic quantification should be interpreted. I begin by noting that when in general talking about a domain of quantification in the context of dual-intuitionistic logic, the domain of quantification which the universal quantifier ranges over is not the domain of quantification which the existential quantifier ranges over. The universal quantifier in particular is idiosyncratic of dual-intuitionistic logic. An informal explication of $\exists$ and $\forall$ is as follows:

$(\exists x)A$ is falsified if for any object, $c$ we may discover, $A[x/c]$ is falsified or it is confirmed that it is not the case that $A[x/c]$.

$(\forall x)A$ is falsified if for some current object, $c$, $A[x/c]$ is falsified.

As is often the case, the possible-world treatment of dual-intuitionistic logic elucidates these notions. So given a collection of observed/discovered objects denoted by $c_1, c_2, ..., c_n$, if $A(c_1)$ and $A(c_2)$ and $A(c_n)$, then $(\forall x)A(x)$. When a universal generalisation 'all $x$ are $A$' is uttered, there is the implicit qualification 'all *observed* $x$ are $A$'. When a 16th century European zoologist uttered 'all swans are white', the universal quantification expressed was that of dual-intuitionistic logic. However strangely enough, in the dual-intuitionistic sense, this utterance would not have entailed that it is not the case that some swan is not white.

## Appendix A

$$\text{Identity and Cut} \quad X, A \vdash A, Y \quad \frac{X \vdash Y, C \qquad C, X \vdash Y}{X \vdash Y}\, Cut$$

$$\text{Negation Rule} \quad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\, \neg R$$

$$\text{Conjunction Rules} \quad \frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\, \wedge L \quad \frac{X \vdash A, Y \qquad X \vdash B, Y}{X \vdash A \wedge B, Y}\, \wedge R$$

$$\text{Disjunction Rules} \quad \frac{X, A \vdash Y \qquad X, B \vdash Y}{X, A \vee B \vdash Y}\, \vee L \quad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\, \vee R$$

$$\text{Structural Rules} \quad \frac{X, A, A \vdash Y}{X, A \vdash Y}\, WL \quad \frac{X \vdash A, A, Y}{X \vdash A, Y}\, WR$$

Figure 14: Sequent rules for Logic of Paradox (LP)

$A \rightarrow B$ can be defined as $\neg A \vee B$ and $A \doteq B$ can be defined as $A \wedge \neg B$. Double Negation Elimination and De Morgan's Laws are taken as definitions, in effect they are rewrite rules:

- $\neg\neg A \Rightarrow A$
- $\neg(A \wedge B) \Rightarrow \neg A \vee \neg B$
- $\neg(A \vee B) \Rightarrow \neg A \wedge \neg B$

These rules should be applied exhaustively first, so that negations only apply to atoms.

Alternatively, we could remove the $\neg R$ rule and add the axiom $\vdash p, \neg p$. This is analogous to the approach taken in [2] for a sequent formalisation of the strong Kleene logic $\mathbf{K}_3$ and corresponds to the tableaux system for **LP** given in ([11], p. 146.).

Quantifiers are added as follows

$$\text{Universal Quantifier Rules} \quad \frac{X, A[x/n] \vdash Y}{X, (\forall x)A \vdash Y}\, \forall L \quad \frac{X \vdash A[x/n], Y}{X \vdash (\forall x)A, Y}\, \forall R\ (m \text{ not in } X, Y)$$

$$\text{Existential Quantifier Rules} \quad \frac{X, A[x/n] \vdash Y}{X, (\exists x)A \vdash Y}\, \exists L\ (m \text{ not in } X, Y) \quad \frac{X \vdash A[x/n], Y}{X \vdash (\exists x)A, Y}\, \exists R$$

With accompanying rewrite rules:

$\neg(\forall x)A \Rightarrow (\exists x)\neg A$

$\neg(\exists x)A \Rightarrow (\forall x)\neg A$

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