Independence-Friendly Quantified Modal Logic

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Abstract

In this paper I look at informational independence in the context of quantified modal logic. Firstly, I define an independence-friendly (IF) constant domain quantified modal logic and to an extent investigate its expressive power relative to standard constant domain quantified modal logic. Secondly, I extend this investigation to variable domain quantified modal logic, where things become more interesting. I am interested in confining investigation to those independence relations particular to quantified modal logic, so the IF quantified modal logics investigated augment standard quantified constant domain modal logic by only allowing $\exists$ quantifiers to be marked as independent from $\Box$ operators and $\Diamond$ operators to be marked as independent from $\forall$ quantifiers. I then discuss some applications/motivations of informational independence in an intensional context. I will use the subscripted equivalence sign $\equiv_t$ to indicate truth equivalence; $A \equiv_t B$ means that A is true iff B is true.

keywords: game-theoretical semantics, independence-friendly logic, first-order modal logic, quantified modal logic

1 Constant Domain Quantified Modal Logic

Let **CK** denote the constant domain quantified modal logic corresponding to the propositional modal logic **K** [1]. The syntax of **CK** augments the language of first-order logic with the operators $\Box$ and $\Diamond$. An interpretation for the language is a quadruple $\langle D, W, R, v \rangle$, where:

- W is a set of worlds
- R is a binary accessibility relation on W
- D is the non-empty domain of quantification
- v assigns each constant, c, of the language a member, $v(c)$, of D and each pair comprising a world, w, and an n-place predicate, P, a member of the truth value set $\{0, 1\}$

The truth conditions for the connectives and modal operators are as in the propositional case. The truth conditions for the quantifiers are as in first-order logic. Thus, for every world, w:

- $v_w((\forall x)A) = 1$ iff for all $d \in D, v_w(A_x(f_d)) = 1$ (otherwise it is 0)
- $v_w((\exists x)A) = 1$ iff for some $d \in D, v_w(A_x(f_d)) = 1$ (otherwise it is 0)

Let **IFCK** denote the language of independence-friendly **CK** and φ be a formula of ordinary **CK** in negation normal form. A formula of **IFCK** is obtained as follows.

Firstly, we index each of the modal operators in φ with a number. It is important to note that we are not dealing with a multiply modal logic; the index numbers are merely used for identification purposes. We define a numbering function, $MI(\varphi, n)$, where $n \in \mathbb{N}$. Starting with $n = 1$, we inductively employ MI to index each of the modal operators of a given **CK** formula in preparation for its usage as an **IFCK** formula. MI is defined as follows:

[1] K is the basic modal logic characterised by the axiom $\Box(A \to B) \to (\Box A \to \Box B)$. The following formulation can be found in [5]

$$
\begin{array}{lll}
MI(p,n) & = & p \\
MI(\neg p,n) & = & \neg p \\
MI(\varphi \wedge \phi, n) & = & MI(\varphi,n) \wedge MI(\phi,n) \\
MI(\varphi \vee \phi, n) & = & MI(\varphi,n) \vee MI(\phi,n) \\
MI(\Box\varphi, n) & = & \Box_n MI(\varphi, n+1) \\
MI(\Diamond\varphi, n) & = & \Diamond_n MI(\varphi, n+1) \\
MI((\forall x)\varphi, n) & = & (\forall x) MI(\varphi, n) \\
MI((\exists x)\varphi, n) & = & (\exists x) MI(\varphi, n)
\end{array}
$$

Two examples of such application of MI:

- $\Box\Diamond\Box(\exists x)P(x)$ becomes $\Box_1\Diamond_2\Box_3(\exists x)P(x)$
- $(\forall x)\Box(\Diamond\Box P(x) \wedge \Diamond(\exists y)F(x,y))$ becomes $(\forall x)\Box_1(\Diamond_2\Box_3 P(x) \wedge \Diamond_2(\exists y)F(x,y))$

Given an indexed formula, a formula of **IFCK** is obtained by any number of the following two steps:

- If a $\Diamond$ occurs in φ within the scope of a number of universal quantifiers which include $(\forall x_1), (\forall x_2), ...$, it may be replaced by $\Diamond_{/\{x_1,x_2,...\}}$.
- If an existential quantifier $(\exists y)$ occurs within φ, it may be replaced by $(\exists y_{/\{n,m,...\}})$, where $n, m, ...$ are numbers used to index instances of the $\Box$ operator which $(\exists y)$ is within the scope of.

The following formulas are members of **IFCK**:

$\Box_1(\exists x_{/\{1\}})P(x)$, $(\forall x)\Diamond_{1/\{x\}}\Diamond_2 P(x)$, $(\forall x)(\forall y)\Diamond_{1/\{x,y\}}P(x,y)$ and $(\forall x)(\Diamond_{1/\{x\}}P(x) \wedge \Diamond_{2/\{x\}}Q(x))$

By contrast, the following formulas are not members of **IFCK**

$\Box_1(\forall x_{/\{1\}})P(x)$, $(\exists x)\Diamond_{1/\{x\}}\Diamond_2 P(x)$, $(\forall x)(\forall y)\Box_{1/\{x,y\}}P(x,y)$ and $\Box_1(\Box_2(\exists x)P(x) \vee (\exists y_{/\{2\}})F(y))$

2 Semantics

The semantics for this logic can basically be obtained by translating IF quantified modal logic formulas into independence-friendly first-order logic **IF-FOL** formulas and then using the standard semantics for IF-FOL. **Note: The semantics are straightforward to those with a basic understanding of IF logic and GTS. I am still working on a clear way to present the semantics in a game-theoretical way.**

I here define a procedure to translate **CK** into **FOL**, to be used in subsequent sections. This translation is an extension of the well-known *standard translation*[2] from propositional modal logic into first-order logic. Although there is no 'standard' translation procedure as such for quantified modal logic, the translation function will be denoted by ST. My approach will be to use a two-sorted first-order logic; one sort for worlds and one sort for individuals. Of course, we can easily translate these two-sorted first-order logic formulas into one-sorted ones. The two-sorted first-order language under consideration here is defined as follows. It is assumed that a countably infinite set of first-order variables for worlds and a countably infinite set of first-order variables for individuals are given. The sets are assumed to be disjoint. In the following definition, the metavariables w_n range over first-order variables for worlds and the metavariable x ranges over first-order variables for individuals. The standard translation ST_{w_n}, where w_n is a constant representing the current world, is defined as follows:

[2] [1]

$$
\begin{array}{lcl}
ST_{w_n}(P(x_1,...,x_n)) & = & P(x_1,...,x_n,w_n) \\
ST_{w_n}(\neg\varphi) & = & \neg ST_{w_n}(\varphi) \\
ST_{w_n}(\varphi\wedge\phi) & = & ST_{w_n}(\varphi)\wedge ST_{w_n}(\phi) \\
ST_{w_n}(\varphi\vee\phi) & = & ST_{w_n}(\varphi)\vee ST_{w_n}(\phi) \\
ST_{w_n}(\Diamond\varphi) & = & (\exists w_{n+1})(R(w_n,w_{n+1})\wedge ST_{w_{n+1}}(\varphi)) \\
ST_{w_n}(\Box\varphi) & = & (\forall w_{n+1})(R(w_n,w_{n+1})\rightarrow ST_{w_{n+1}}(\varphi)) \\
ST_{w_n}((\exists x)\varphi) & = & (\exists x)ST_{w_n}(\varphi) \\
ST_{w_n}((\forall x)\varphi) & = & (\forall x)ST_{w_n}(\varphi)
\end{array}
$$

Note that the definition of $ST_{w_{n+1}}$ is obtained by exchanging w_{n+1} for w_n. Also, a propositional atom p is considered a zero-place predicate, so $ST_{w_n}(p) = P(w_n)$.

To state formally that the translation given above is truth-preserving with respect to constant domain semantics, it is noted that a constant domain model for first-order modal logic can be considered as a model for two-sorted first-order logic and vice versa.

Definition Given a **CK** model $\mathcal{M} = \langle D, W, R, v\rangle$ for first-order modal logic, a model $\mathcal{M}^* = \langle D^*, W^*, v^*\rangle$ for two-sorted first-order logic is defined by letting

- $v^*(R) = R$
- $v^*P((d_1,...,d_n,w)) = 1$ iff $v_wP(d_1,...,d_n) = 1$
- $D = D^*$
- $W = W^*$

This translation procedure is truth-preserving: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}^* \models ST_w(\varphi)$.[3] Thus, first-order modal logic, considered as a language for talking about constant domain models, has the same expressive power as the fragment of two-sorted first-order logic obtained by taking the image of first-order modal logic under the translation ST_w.

This translation procedure can be simply adjusted to give a translation of **IFCK** into **IF-FOL**. The translation ST_{w_n} defined above be extended with the following:

$$
\begin{array}{lcl}
ST_{w_n}(\Diamond_{/\{x_1,\ldots,x_n\}}\varphi) & = & (\exists w_{n+1/\{x_1,\ldots,x_n\}})(R(w_n,w_{n+1})\wedge ST_{w_{n+1}}(\varphi)) \\
ST_{w_n}((\exists x_{/\{1,\ldots,m\}})\varphi) & = & (\exists x_{/\{w_1,\ldots,w_m\}})ST_{w_n}(\varphi)
\end{array}
$$

3 Some Observations on the Expressive Power of IFCK

Fact 3.1 *The **IFCK** formulas $(\forall x)\Diamond_{1/\{x\}}P(x)$ and $\Box_1(\exists x_{/\{1\}})P(x)$ can be translated into equivalent **CK** formulas:*

$$(\forall x)\Diamond_{1/\{x\}}P(x) \equiv_t \Diamond(\forall x)P(x) \text{ and } \Box_1(\exists x_{/\{1\}})P(x) \equiv_t (\exists x)\Box P(x)$$

Proof These equivalences are straightforward. Here is a demonstration of the latter equivalence via translation of each formula to a **FOL** formula. Let $\phi_1 = (\exists x)\Box P(x)$ and $\phi_2 = \Box_1(\exists x_{/\{1\}})P(x)$

$$
\begin{aligned}
ST_{w_0}(\phi_1) &= (\exists x)ST_{w_0}(\Box P(x)) \\
&= (\exists x)(\forall w_1)(R(w_0,w_1)\rightarrow ST_{w_1}(P(x))) \\
&= (\exists x)(\forall w_1)(R(w_0,w_1)\rightarrow P(x,w_1))
\end{aligned}
$$

$$
\begin{aligned}
ST_{w_0}(\phi_2) &= (\forall w_1)(R(w_0,w_1)\rightarrow ST_{w_1}((\exists x_{/\{w_1\}})P(x))) \\
&= (\forall w_1)(R(w_0,w_1)\rightarrow (\exists x_{/\{w_1\}})ST_{w_1}(P(x))) \\
&= (\forall w_1)(R(w_0,w_1)\rightarrow (\exists x_{/\{w_1\}})P(x,w_1)) \\
&= (\exists x)(\forall w_1)(R(w_0,w_1)\rightarrow P(x,w_1))
\end{aligned}
$$

$\therefore ST_{w_0}(\phi_1) = ST_{w_0}(\phi_2)$ ■

[3] More on this formal statement can be found in the *First-Order Modal Logic* chapter of the recently released *Handbook of Modal Logic*, published by Elsevier in 2006

Fact 3.2 *The **IFCK** formulas* $(\forall x)\Box_1\Diamond_{2/\{x\}}P(x)$ *and* $\Box_1(\forall x)(\exists y_{/\{1\}})P(x,y)$ *can be translated into equivalent **CK** formulas:*

$$(\forall x)\Box_1\Diamond_{2/\{x\}}P(x) \equiv_t \Box\Diamond(\forall x)P(x) \text{ and } \Box_1(\forall x)(\exists y_{/\{1\}})P(x,y) \equiv_t (\forall x)(\exists y)\Box P(x,y)$$

Proof This relies on the fact that in **CK** $(\forall x)\Box A \equiv \Box(\forall x)A$, so we can syntactically swap the $(\forall x)$ and $\Box$. Here is a demonstration of the latter equivalence. Let $\phi_1 = (\forall x)(\exists y)\Box_1 P(x,y)$ and $\phi_2 = \Box_1(\forall x)(\exists y_{/\{1\}})P(x,y)$.

$$\begin{aligned}
ST_{w_0}(\phi_1) &= (\forall x)ST_{w_0}((\exists y)\Box_1 P(x,y))\\
&= (\forall x)(\exists y)ST_{w_0}(\Box_1 P(x,y))\\
&= (\forall x)(\exists y)(\forall w_1)(R(w_0,w_1) \rightarrow ST_{w_1}(P(x,y)))\\
&= (\forall x)(\exists y)(\forall w_1)(R(w_0,w_1) \rightarrow P(x,y,w_1))
\end{aligned}$$

$$\begin{aligned}
ST_{w_0}(\phi_2) &= (\forall w_1)(R(w_0,w_1) \rightarrow ST_{w_1}((\forall x)(\exists y_{/w_1})P(x,y)))\\
&= (\forall w_1)(R(w_0,w_1) \rightarrow (\forall x)ST_{w_1}((\exists y_{/w_1})P(x,y)))\\
&= (\forall w_1)(R(w_0,w_1) \rightarrow (\forall x)(\exists y_{/w_1})ST_{w_1}(P(x,y)))\\
&= (\forall w_1)(R(w_0,w_1) \rightarrow (\forall x)(\exists y_{/w_1})P(x,y,w_1))\\
&= (\forall w_1)(\forall x)(\exists y_{/w_1})(R(w_0,w_1) \rightarrow P(x,y,w_1))\\
&= (\exists y)(\forall w_1)(\forall x)(R(w_0,w_1) \rightarrow P(x,y,w_1))
\end{aligned}$$

$\therefore ST_{w_0}(\phi_1) = ST_{w_0}(\phi_2)$ ■

Fact 3.3 *The **IFCK** formula* $\Box_1\Box_2(\exists x_{/\{2\}})(\exists y_{/\{1\}})P(x,y)$ *can be translated into an equivalent **CK** formula.*

Proof This formula can be considered as an analogue of the basic genuine **IF-FOL** prefix $(\forall x)(\forall y)(\exists z_{/\{y\}})(\exists u_{/\{x\}})$. Firstly, $\Box_1\Box_2(\exists x_{/\{2\}})(\exists y_{/\{1\}})P(x,y)$ can straightforwardly be translated to

$$\Box_1(\exists x)\Box_2(\exists y_{/\{1\}})P(x,y) \tag{3.1}$$

Secondly, we can eliminate the remaining slash to get:

$$\Box_1(\exists x)\Box_2(\exists y)P(x,y) \tag{3.2}$$

This second move, of simply removing the slash which indicates that $(\forall x)$ is independent of $\Box_1$ requires justification. To see why we can do this, consider the formula:

$$\varphi = \Box_1\Box_2(\exists x_{/\{1\}})P(x)$$

For a model $\mathcal{M}$ and world w_0, the game $G_A(\varphi, \mathcal{M}, w_0)$ will be such that all histories at which Verifier is to make a selection for $(\exists x_{/\{1\}})$ will have a world sequence of the form:

$$(w_0, w_i, w_j)$$

where $j,k \in \mathbb{N}$. Now, all histories h_1 and h_2 such that $h_1 \sim_{\mathcal{V}} h_2$ will be of the form (w_0, w_x, w_n), where $x, n \in \mathbb{N}$, x is a variable relative to the histories and n is a constant relative to the histories, so that w_0 and w_n are worlds common to both histories and w_x is not common to both histories. For Verifier to have a winning and uniform strategy, it must be the case that $f_{\mathcal{V}}(h_1) = f_{\mathcal{V}}(h_2)$. Now, since both of these histories in the same information partition end at the same world, it trivially follows that if both histories are winning then the same selection for $(\exists x_{/\{1\}})$ can be made. This exemplifies the fact that, as pointed out by Tero Tulenheimo, modal logic is transitional; each transition made when evaluating a modal operator is guarded by an accessibility relation and depends only on where the previous transition led. Hence, the semantics of **IFCK** precludes the possibility of having the effect of the Henkin quantifier in this case, where two $\Box$ operators replace the two $(\forall)$ quantifiers. ■

Fact 3.4 *The* ***IFCK*** *formula* $(\forall x)(\forall y)\Diamond_{1/\{y\}}\Diamond_{2/\{x\}}P(x,y)$ *can be translated into an equivalent* ***CK*** *formula* ***[Note: This fact might be incorrect I think. I need to think about the proof more]****.*

Proof This is the other basic **IFCK** analogue of the Henkin Quantifier. Firstly, $(\forall x)(\forall y)\Diamond_{1/\{y\}}\Diamond_{2/\{x\}}P(x,y)$ can be straightforwardly translated to:

$$(\forall x)\Diamond_1(\forall y)\Diamond_{2/\{x\}}P(x,y) \tag{3.3}$$

Let φ stand for this formula. Translating this formula to **IF-FOL**, we get:

$$\begin{aligned}
ST_{w_0}(\varphi) &= (\forall x)ST_{w_0}(\Diamond_1(\forall y)\Diamond_2/\{x\}P(x,y)) \\
&= (\forall x)(\exists w_1)(R(w_0,w_1) \wedge ST_{w_1}((\forall y)\Diamond_2/\{x\}P(x,y)) \\
&= (\forall x)(\exists w_1)(R(w_0,w_1) \wedge (\forall y)ST_{w_1}(\Diamond_2/\{x\}P(x,y)) \\
&= (\forall x)(\exists w_1)(R(w_0,w_1) \wedge (\forall y)(\exists w_{2/\{x\}})(R(w_1,w_2) \wedge ST_{w_2}(P(x,y))) \\
&= (\forall x)(\exists w_1)(R(w_0,w_1) \wedge (\forall y)(\exists w_{2/\{x\}})(R(w_1,w_2) \wedge P(x,y,w_2))) \\
&= (\forall x)(\exists w_1)(\forall y)(\exists w_{2/\{x\}})(R(w_0,w_1) \wedge (R(w_1,w_2) \wedge P(x,y,w_2)))
\end{aligned}$$

The prefix of this formula expresses partially ordered quantification. But is the slash necessary? It is clear that the $(\exists w_2)$ has to be within the scope of the $(\forall y)$ and out of the scope of the $(\forall x)$. What about the $(\exists w_1)$? Must it remain within the scope of the $(\forall x)$? The answer is no. A justification for this assertion is as follows. Consider the following formula, which is derived by removing the $(\forall y)$ and the instances of y from $ST_{w_0}(\varphi)$.

$$(\forall x)(\exists w_1)(\exists w_{2/\{x\}})(R(w_0,w_1) \wedge (R(w_1,w_2) \wedge P(x,w_2)))$$

Now, it is clear that the $(\exists w_2)$ must be out of the scope of the $(\forall x)$. But is there a difference between the following two formulas, one with the $(\exists w_1)$ within the scope of the $(\forall x)$ and the other with the $(\exists w_1)$ out of the scope of the $(\forall x)$:

$$(\exists w_2)(\forall x)(\exists w_1)(R(w_0,w_1) \wedge R(w_1,w_2) \wedge P(x,w_2)) \tag{3.4}$$

and

$$(\exists w_1)(\exists w_2)(\forall x)(R(w_0,w_1) \wedge R(w_1,w_2) \wedge P(x,w_2)) \tag{3.5}$$

The answer is no. That is, (3.4) is equivalent with (3.5).[4]. For this reason, we can translate $ST_{w_0}(\varphi)$ to

$$(\exists w_1)(\forall y)(\exists w_2)(\forall x)(R(w_0,w_1) \wedge (R(w_1,w_2) \wedge P(x,y,w_2)))$$

which can be reverse translated into the **CK** formula

$$\Diamond(\forall y)\Diamond(\forall x)P(x,y).$$

Therefore

$$(\forall x)(\forall y)\Diamond_{1/\{y\}}\Diamond_{2/\{x\}}P(x,y) \equiv_t \Diamond(\forall y)\Diamond(\forall x)P(x,y).$$

■

[4]Automated Theorem Prover Otter used

Fact 3.5 *The* ***IFCK*** *formula* $(\forall x)(\exists y)\Box_1\Diamond_{2/\{x\}}P(x,y)$ *can not be translated into an equivalent* ***CK*** *formula.*

Proof Let $\varphi = (\forall x)(\exists y)\Box_1\Diamond_2/\{x\}P(x,y)$.

$$\begin{aligned} ST_{w_0}(\varphi) &= (\forall x)(\exists y)(ST_{w_0}(\Box_1\Diamond_{2/\{x\}}P(x,y))) \\ &= (\forall x)(\exists y)(\forall w_1)(R(w_0,w_1) \to ST_{w_1}(\Diamond_{2/\{x\}}P(x,y))) \\ &= (\forall x)(\exists y)(\forall w_1)(R(w_0,w_1) \to (\exists w_{2/\{x\}})(R(w_1,w_2) \wedge ST_{w_2}(P(x,y)))) \\ &= (\forall x)(\exists y)(\forall w_1)(R(w_0,w_1) \to (\exists w_{2/\{x\}})(R(w_1,w_2) \wedge P(x,y,w_2))) \\ &= (\forall x)(\exists y)(\forall w_1)(\exists w_{2/\{x\}})(R(w_0,w_1) \to (R(w_1,w_2) \wedge P(x,y,w_2))) \end{aligned}$$

It is clear that the $(\exists w_2)$ must be out of the scope of the $(\forall x)$ and in the scope of the $(\forall w_1)$. The $(\exists y)$ must be in the scope of the $(\forall x)$. Must it be out of the scope of the $(\forall w_1)$? The answer is yes. A justification for this assertion is as follows. Consider the formula

$$(\forall x)(\exists y)(\forall w_1)(\exists w_2)(R(w_0,w_1) \to (R(w_1,w_2) \wedge P(x,y,w_2)))$$

which is simply the translation of

$$(\forall x)(\exists y)\Box\Diamond P(x,y)$$

Now, is it the same as the following formula, which is simply the result of swapping the $(\exists y)$ and $(\forall w_1)$

$$(\forall x)(\forall w_1)(\exists y)(\exists w_2)(R(w_0,w_1) \to (R(w_1,w_2) \wedge P(x,y,w_2)))$$

No, the two formulas are not equivalent[5], so the $(\exists y)$ must be out of the scope of the $(\forall w_1)$. Therefore, $ST_{w_0}(\varphi)$ is an **IF-FOL** formula which exceeds the expressive power of **FOL** and can not be translated into an equivalent **CK** formula.

■

Fact 3.6 *The* ***IFCK*** *formula* $(\forall x)(\exists y)(\forall z)\Diamond_{/\{x\}}P(x,y,z)$ *can not be translated into an equivalent* ***FOL*** *formula.*

Proof Let $\varphi = (\forall x)(\exists y)(\forall z)\Diamond_{/\{x\}}P(x,y,z)$

$$\begin{aligned} ST_{w_0}(\varphi) &= (\forall x)(\exists y)(\forall z)ST_{w_0}(\Diamond_{/\{x\}}P(x,y,z)) \\ &= (\forall x)(\exists y)(\forall z)(\exists w_{1/\{x\}})(R(w_0,w_1) \wedge ST_{w_1}(P(x,y,z))) \\ &= (\forall x)(\exists y)(\forall z)(\exists w_{1/\{x\}})(R(w_0,w_1) \wedge P(x,y,z,w_1)) \end{aligned}$$

It is clear that $ST_{w_0}(\varphi)$ is a genuine IF first-order logic formula. ■

Corollary 3.1 *The* ***IFCK*** *formula* $(\forall x)(\exists y)(\forall z)\Diamond_{/\{x\}}P(x,y,z)$ *can not be translated into an equivalent* ***CK*** *formula.*

Fact 3.7 *The* ***IFCK*** *formula* $\Box_1(\exists x)(\forall y)(\exists z_{/\{1\}})P(x,y,z)$ *can not be translated into an equivalent* ***FOL*** *formula.*

[5] Automated Theorem Prover Otter used

Proof Let $\varphi = \Box_1(\exists x)(\forall y)(\exists z_{/\{1\}})P(x,y,z)$

$$\begin{aligned} ST_{w_0}(\varphi) &= (\forall w_1)(R(w_0,w_1) \to ST_{w_1}((\exists x)(\forall y)(\exists z_{/\{1\}})P(x,y,z))) \\ &= (\forall w_1)(R(w_0,w_1) \to (\exists x)(\forall y)(\exists z_{/\{1\}})P(x,y,z,w_1)) \end{aligned}$$

It is clear that $ST_{w_0}(\varphi)$ is a genuine IF first-order logic formula. ■

Corollary 3.2 *The **IFCK** formula $\Box_1(\exists x)(\forall y)(\exists z_{/\{1\}})P(x,y,z)$ can not be translated into an equivalent **CK** formula.*

Remark **IFCK** has the same expressive power as the fragment of IF two-sorted logic obtained by taking the image of **IFCK** under the translation ST_w defined in Section (2). This fragment is more expressive than **FOL** as evidenced by certain facts above, but also obviously can not express simple **IF-FOL** formulas such as

$$(\forall x)(\exists y)(\forall z)(\exists u/_{\{x\}})(F(x,y,z,u))$$

Remark I have as yet found with definitive certainty an **IFCK** formula which has a **FOL** equivalent but not a **CK** equivalent. I strongly suspect that the **IFCK** formula

$$\varphi = (\forall x)(A(x) \vee \Box_1 \Diamond_{2/\{x\}} B(x))$$

is one such formula. It is equivalent to the **FOL** formula

$$(\forall w_1)(\exists w_2)(\forall x)(A(x,w_0) \vee (R(w_0,w_1) \to ((R(w_1,w_2) \wedge B(x,w_2))))) \tag{3.6}$$

(3.6) can not be reverse-translated (RT) into a **CK** formula via the standard translation. Such a translation, $RT(ST_{w_0}(\varphi))$, would convert the $(\forall w_1)$ to the operator $\Box_1$. However, this operator must only bind the right disjunct, which is equivalent to moving the $(\forall w_1)$ inwards so that it only binds the right disjunct. However, doing so would violate the required scope ordering of $(\forall w_1)(\exists w_2)(\forall x)$

It also seems that there is no **CK** formula θ such that $(\forall x)(A(x) \vee \Box\Diamond B(x)) \wedge \theta \equiv_t (\forall x)(A(x) \vee \Box_1 \Diamond_{2/\{x\}} B(x))$.

4 Variable Domain Quantified Modal Logic

I now come to a briefer, though more interesting look at employment of the slash indicator in the context of variable domain quantified modal logic.

Let **VK** / **IFVK** denote the variable domain versions of **CK** / **IFCK**. **IFVK** is exactly the same as **IFCK** with the exception that for every $w \in W$, v maps w to a subset of D, that is, $v(w) \subseteq D$. $v(w)$ is the domain at world w. I will write it as D_w.

The truth conditions for the quantifers are:

- $v_w((\forall x)A) = 1$ iff for all $d \in D_w, v_w(A_x(c_d)) = 1$ (otherwise it is 0)
- $v_w((\exists x)A) = 1$ iff for some $d \in D_w, v_w(A_x(c_d)) = 1$ (otherwise it is 0)

Correspondingly, the game rules are adjusted so that when a player is to select a domain object, they must choose from objects within the domain of the world which is the current state of the game. The first-order translation of **IFVK** is the same as that given in Section 2 except that the clauses for the quantifiers are replaced by the following, where E is the special existence predicate:

$$\begin{array}{lcl} ST_w((\forall x)\varphi) & = & (\forall x)(E(w,x) \rightarrow ST_w(\varphi)) \\ ST_w((\exists x)\varphi) & = & (\exists x)(E(w,x) \wedge ST_w(\varphi)) \\ ST_w((\exists x_{/\{1,\ldots,n\}})\varphi) & = & (\exists x_{/\{w_1,\ldots,w_n\}})(E(w,x) \wedge ST_w(\varphi)) \end{array}$$

Fact 3.1 stated that

$$(\forall x)\Diamond_{1/\{x\}}P(x) \equiv_t \Diamond(\forall x)P(x) \text{ and } \Box_1(\exists x_{/\{1\}})P(x) \equiv_t (\exists x)\Box P(x)$$

Let us see how these equivalences fare in **IFVK**.

Fact 4.1 *The equivalence* $(\forall x)\Diamond_{1/\{x\}}P(x) \equiv_t \Diamond(\forall x)P(x)$ *does not hold in **IFVK**.*

Proof Let $\varphi_1 = \Diamond(\forall x)P(x)$ and $\varphi_2 = (\forall x)\Diamond_{/\{x\}}P(x)$

$$\begin{aligned} ST_{w_0}(\varphi_1) &= (\exists w_1)(R(w_0,w_1) \wedge ST_{w_1}((\forall x)P(x))) \\ &= (\exists w_1)(R(w_0,w_1) \wedge (\forall x)(E(w_1,x) \rightarrow ST_{w_1}(P(x)))) \\ &= (\exists w_1)(R(w_0,w_1) \wedge (\forall x)(E(w_1,x) \rightarrow P(x,w_1))) \\ &= (\exists w_1)(\forall x)(R(w_0,w_1) \wedge (E(w_1,x) \rightarrow P(x,w_1))) \end{aligned}$$

$$\begin{aligned} ST_{w_0}(\varphi_2) &= (\forall x)(E(w_0,x) \rightarrow ST_{w_0}(\Diamond_{/\{x\}}P(x))) \\ &= (\forall x)(E(w_0,x) \rightarrow (\exists w_{1/\{x\}})(R(w_0,w_1) \wedge ST_{w_1}(P(x)))) \\ &= (\forall x)(E(w_0,x) \rightarrow (\exists w_{1/\{x\}})(R(w_0,w_1) \wedge P(x,w_1))) \\ &= (\exists w_1)(\forall x)(E(w_0,x) \rightarrow (R(w_0,w_1) \wedge P(x,w_1))) \end{aligned}$$

$\therefore ST_{w_0}(\varphi_1) \neq ST_{w_0}(\varphi_2)$. In fact, neither of the arguments $(\forall x)\Diamond_{1/\{x\}}P(x) \vdash \Diamond(\forall x)P(x)$ and $\Diamond(\forall x)P(x) \vdash (\forall x)\Diamond_{1/\{x\}}P(x)$ are valid in **IFVK**. ■

It is clear that without this equivalence, the formula $(\forall x)\Diamond_{1/\{x\}}P(x)$ is not reducible to a formula in **IFVK**.

Fact 4.2 *The equivalence* $\Box_1(\exists x_{/\{1\}})P(x) \equiv_t (\exists x)\Box P(x)$ *does not hold in **IFVK**.*

Proof Let $\varphi_1 = (\exists x)\Box P(x)$ and $\varphi_2 = \Box_1(\exists x_{/\{1\}})P(x)$

$$\begin{aligned} ST_{w_0}(\varphi_1 &= (\exists x)(E(w_0,x) \wedge ST_{w_0}(\Box P(x))) \\ &= (\exists x)(E(w_0,x) \wedge (\forall w_1)(R(w_0,w_1) \rightarrow ST_{w_1}(P(x)))) \\ &= (\exists x)(E(w_0,x) \wedge (\forall w_1)(R(w_0,w_1) \rightarrow P(x,w_1))) \\ &= (\exists x)(\forall w_1)(E(w_0,x) \wedge (R(w_0,w_1) \rightarrow P(x,w_1))) \end{aligned}$$

$$\begin{aligned} ST_{w_0}(\varphi_1) &= (\forall w_1)(R(w_0,w_1) \rightarrow ST_{w_1}((\exists x_{/\{1\}})P(x))) \\ &= (\forall w_1)(R(w_0,w_1) \rightarrow (\exists x_{/\{1\}})(E(w_1,x) \wedge ST_{w_1}(P(x)))) \\ &= (\forall w_1)(R(w_0,w_1) \rightarrow (\exists x_{/\{1\}})(E(w_1,x) \wedge P(x,w_1))) \\ &= (\exists x)(\forall w_1)(R(w_0,w_1) \rightarrow (E(w_1,x) \wedge P(x,w_1))) \end{aligned}$$

$\therefore ST_{w_0}((\exists x)\Box P(x)) \neq ST_{w_0}(\Box_1(\exists x_{/\{1\}})P(x))$. In fact, neither of the arguments $\Box_1(\exists x_{/\{1\}})P(x) \vdash (\exists x)\Box P(x)$ and $(\exists x)\Box P(x) \vdash \Box_1(\exists x_{/\{1\}})P(x)$ are valid in **IFVK** ■

It is clear that without this equivalence, the formula $\Box_1(\exists x_{/\{1\}})P(x)$ is not reducible to a formula in **IFVK**.

Failure of these basic equivalences emphasises the significance of employing the slash independence indicator in the context of variable domain quantified modal logic. In the case of $(\forall x)\Diamond_{1/\{x\}}P(x)$, since our domain might increase, the $\Diamond$ cannot precede the $(\forall x)$. However we still might want a way of saying that there is a *particular* world which the current world accesses such that the property P holds at the accessed world for every object in the *actual* world.

In the case of $\Box_1(\exists x_{/\{1\}})P(x)$, the slash expresses the restriction that although the domains between worlds may vary, there must be at least one object common to the domains of each world accessed by the current world for which P holds at each world accessed. As an extreme example, consider the difference between the following two:

$$(\forall x)x \neq x \wedge (\exists x)\Box A(x) \vdash \text{ and} (\forall) x \neq x \wedge \Box_1(\exists x_{/\{1\}}A(x) \nvdash \tag{4.1}$$

Since these two basic equivalences do not hold, it follows that equivalences observed with regard to **IFCK** do not hold in **IFVK**. These arguments can however be made valid by adding strengthening conditions to **IFVK**:

Definition Domain increasing condition: if wRw' then $D_w \subseteq D_{w'}$

Definition Domain decreasing condition: if wRw' then $D_{w'} \subseteq D_w$

Definition Frame Reflexivity: $(\forall x)R(x,x)$

Definition Negativity Constraint: If $v(P(d_1,...,d_n)) = 1$ then $v(E(d_1)) = 1$,...,$v(E(d_n)) = 1$.

Fact 4.3 *Given the domain increasing condition, the argument* $\Diamond(\forall x)P(x) \vdash (\forall x)\Diamond_{1/\{x\}}P(x)$ *is valid in* ***IFVK****.*

Fact 4.4 *Given the domain increasing condition, the argument* $(\exists x)\Box P(x) \vdash \Box_1(\exists x_{/\{1\}})P(x)$ *is valid in* ***IFVK****.*

Fact 4.5 *Given the domain decreasing condition, the argument* $(\forall x)\Diamond_{1/\{x\}}P(x) \vdash \Diamond(\forall x)P(x)$ *is valid in* ***IFVK****.*

Fact 4.6 *Given the domain decreasing condition, the argument* $\Box_1(\exists x_{/\{1\}})P(x) \vdash (\exists x)\Box P(x)$ *is valid in* ***IFVK****.*

Fact 4.7 *Given the reflexivity constraint, the argument* $\Box_1(\exists x_{/\{1\}})P(x) \vdash (\exists x)\Box P(x)$ *is valid in* ***IFVK****.*

Fact 4.8 *Given the negativity constraint, the argument* $(\exists x)\Box P(x) \vdash \Box_1(\exists x_{/\{1\}})P(x)$ *is valid in* ***IFVK****.*

Thus the addition of certain variable-domain preserving collections of hybrid constraints can have an effect on the validity of some basic equivalences. For example, given the collection of the domain increasing and reflexivity constraint, the basic equivalence $\Box_1(\exists x_{/\{1\}})P(x) \equiv_t (\exists x)\Box P(x)$ holds.

Despite this, there remain equivalences observed in Section 3 which would still not hold given the availability of this equivalence. For example, the equivalences observed in Fact 3.2 rely on the equivalence $(\forall x)\Box A \equiv \Box(\forall x)A$, which fails unless domain of quantification is both decreasing and increasing.

Also, note that whilst the following two arguments are valid in **IFVK**

$\Box(\exists x)\neg P(x), \Diamond(\forall x)P(x) \vdash_{IFVK}$
$(\forall x)\Diamond\neg P(x), (\exists x)\Box P(x) \vdash_{IFVK}$

The following two counterparts are not

$\Box(\exists x)\neg P(x), (\forall x)\Diamond_{/\{x\}}P(x) \nvdash_{IFVK}$
$(\forall x)\Diamond\neg P(x), \Box_1(\exists x_{/\{1\}})P(x) \nvdash_{IFVK}$

5 Applications/Motivations

In this section, feeding off results of the last section, I take a look at what can be done with informational independence in intensional contexts. In general, the incorporation of informational independence raises new possibilities with regard to symbolising and analysing natural language statements and tackling philosophical issues. In discussion that follows, as well as considering examples involving the alethic modal operators ($\Box$ and $\Diamond$), I will be considering epistemic, doxastic and temporal logic examples as well. Before continuing, I will now briefly clarify the terminology used in such logics. With temporal logic, the two counterparts to the $\Box$ operator are the operators $[F]$ and $[P]$, which stand for 'It is always going to be the case that' and 'It always has been the case that' respectively. The two counterparts to the $\Diamond$ operator are the operators $\langle F\rangle$ and $\langle P\rangle$, which stand for 'At some time in the future it will be the case that' and 'At some time in the past it has been the case that' respectively. The epistemic counterpart of $\Box$ is the operator

K_a, which stands for 'a knows that'. The doxastic counterpart of $\Box$ is the operator B_a, which stands for 'a believes that'. The $\Diamond$ operator counterparts for epistemic and doxastic logic are not as standard. I will here take the epistemic counterpart to be the operator C_a, which stands for 'a finds it credible that', and the doxastic counterpart to be the operator P_a, which stands for 'a finds it plausible that'. In examples where it is not required, modal operators are not indexed. Also, although I do not specify a particular set of frame conditions for these logics in discussion, I am mindful of the frame conditions which are associated with each logic.

5.1

An important distinction in philosophy is the *De Re / De Dicto* distinction. There are two different conceptions of this distinction which I am interested in.

Consider the following sentence, derived from an example due to Quine:

$$\text{George believes that someone is a spy} \tag{5.1}$$

(5.1) is ambiguous in at least two ways. On one interpretation, (5.1) says that

$$\text{George believes that there are spies} \tag{5.2}$$

Interpreted in this way, (5.1) does not entail that George has belief about any particular persons being a spy. On the other interpretation of (5.1), George has not just the general belief that there are spies, but believes of some particular person or persons, that they are spies. Interpreted in this way, (5.1) says something like

$$\text{There is someone who George believes is a spy} \tag{5.3}$$

With quantified modal logic, the distinction between (5.2) and (5.3) can be seen as a distinction of scope for the existential quantifier involved. The logical formula representing the interpretation captured by (5.2) (referred to as the *de dicto* reading) has the existential quantifier and the variable it binds occurring as constituents of the relevant 'believes that' clause. In that case, the quantifier is said to have *narrow scope* relative to the belief operator. With the logical formula representing the interpretation captured by (5.3) (referred to as the *de re* reading), the quantifier occurs outside the scope of the relevant that-clause. On the *de re* reading, the quantifier is said to have *wide scope* relative to the belief operator.

Where B_G stands for 'George believes that' and $S(x)$ stands for 'x is a spy', both (5.2) and (5.3) can be respectively represented as

$$\mathrm{B}_G(\exists x)S(x) \tag{5.4}$$

and

$$(\exists x)\mathrm{B}_G S(x) \tag{5.5}$$

(5.4) rightly does not entail (5.5). Even if George believes that there are spies, it is clear that it does not follow that he believes of anyone in particular that they are a spy. However, the converse implication rightly holds. If George believes of someone in particular that they are a spy, then George believes there exists at least one practitioner of the spying profession.

This idea also applies to alethic discourse. If it is necessary that something is good ($\Box(\exists x)G(x)$) it does not follow that something is necessarily good ($(\exists x)\Box G(x)$). However, if something is necessarily good, then it is necessary that something is good.

This distinction also applies to the distinction between $(\forall x)\Diamond A(x)$ and $\Diamond(\forall x)A(x)$. $(\forall x)\Diamond A(x) \rightarrow \Diamond(\forall x)A(x)$ says something like 'if everything has the potential to be A, its possible for everything to be A', which will not always be the case. $\Diamond(\forall x)A(x) \rightarrow (\forall x)\Diamond A(x)$ says something like 'if it is possible for everything to be A, everything has the potential to be A', which does hold, as it should. The epistemic/doxastic analogues of this example are less perspicuous. For example, 'it is plausible that everything is A', and 'everything is plausibly A', logically mean different things. The former implies the latter but the latter does not imply the former. Grasping this difference is facilitated by considering the difference between their logical representations.

In general, statements are *de re* when they have quantifiers having scope over modal operators and *de dicto* when they have modal operators having scope over quantifiers. In the examples just given, this syntactic difference resulted in one of the model definability. But the *De Re / De Dicto* distinction need not relate solely to this fact, as will now be discussed.

The Barcan Formula $(\forall x)\Box A(x) \rightarrow \Box(\forall x)A(x)$ and the Converse Barcan Formula $\Box(\forall x)A(x) \rightarrow (\forall x)\Box A(x)$ are both valid in **CK**. Despite this equivalence between $(\forall x)\Box A(x)$ and $\Box(\forall x)A(x)$, in a sense they mean two different things. The former says that everything is such that it is necessarily A. It attributes, that is, a necessary or essential property to each individual thing. This is made clearer, by starting with $(\forall x)\Box A(x)$, and substituting a name, a, for the variable x to get $\Box A(a)$; we can see this as saying that "the object denoted by a has the property of necessarily being A". The modality is attached to the object, a, hence the *de re* (which in Latin means 'about the thing'). The latter says that it is necessary that everything is A. It says, in other words, that $(\forall x)A(x)$ is a necessary truth. The modality is attached to the proposition, $(\forall x)A(x)$, hence the *de dicto* (which in Latin means 'about the proposition'). Of course, with variable domain modal logic, neither the Barcan nor Converse Barcan Formulas hold, so the distinction becomes important not only from a semantic/metaphysical perspective but from a model defining one also, which as many have argued, should be the case.

5.2

The notion of informational independence offers an alternative treatment of the *de dicto / de re* distinction without appeal to the notion of scope. Although by no means a threat to the utility of the notion of relative quantifier scope in general, it is worth considering the alternative treatement of the distinction via independence to see as to whether it has any advantages over the traditional treatment via scope.

Take epistemic logic as a representative case. To recap, as far as simple IF quantified modal logic formulas involving a K and a $\exists$ go, there are three possible permutations:

$$\mathrm{K}_a(\exists x)S(x) \tag{5.6}$$

$$(\exists x)\ \mathrm{K}_a S(x) \tag{5.7}$$

$$\mathrm{K}_a(\exists x_{/\{K\}})S(x) \tag{5.8}$$

By now it has been established why (5.6) is the logical form of

a knows that there are spies

and why (5.7) is the logical form of

someone is such that a knows that they are a spy

Although in the case of **IFCK** (5.7) and (5.8) are equivalent, (5.8) may be preferred over (5.7) for symbolising certain knowledge statements. For example, in order to specify the semantics of *knows that* statements, the *de dicto* formula (5.6) is correct. In order to specify the semantics of *knows + wh-word* constructions such as

$$\text{It is known who is a spy} \tag{5.9}$$

although (5.7) and (5.8) are model-theoretically equivalent, its syntactic structure perhaps makes (5.8) preferable and captures the independence implicit in (5.9).

5.3

In the case of variable domain quantified modal logic, the notion of informational independence becomes more significant. As we saw in section 4, it is not the case that the equivalences

$$(\forall x)\Diamond_{1/\{x\}}P(x) \equiv_t \Diamond(\forall x)P(x) \text{ and } \Box_1(\exists x_{/\{1\}})P(x) \equiv_t (\exists x)\Box P(x)$$

hold in **IFVK**.

Here are some examples demonstrating the utility of the slash independence indicator in the context of variable domain quantified modal logic.

Firstly, there are reasonable grounds for countenancing a variable domain of quantification in doxastic contexts. Take the following example.([2], p. 166.) Consider the statement:

a believes that Holmes exists.

This translates easily to:

$$\mathrm{B}_a(\exists x)(\mathrm{Holmes} = x) \tag{5.10}$$

Similarly:

Although Holmes does not exist, a believes that Holmes does exist. (5.11)

This translates easily to:

$$\neg(\exists x)(\mathrm{Holmes} = x) \wedge \mathrm{B}_a(\exists x)(\mathrm{Holmes} = x) \tag{5.12}$$

Both of these statements are perfectly meaningful. However, the employment of constant domain quantified modal logic to analyse these statements is problematic. The translation (5.12) would be contradictory, simply because the left conjunct is a contradiction. Similarly, if we used constant quantified modal logic, then (5.10) would attribute a trivial belief to a because the content of their belief would be a logical truth. This example concerning issues of existential import provide, I believe, justification for the employment of a variable domain doxastic logic.[6]

(5.12) clearly expresses the *de dicto* reading of (5.11). What is the logical form of a *de re* version of (5.11), such as the following?

Although Holmes does not exist, there is someone whom a believes to be Holmes (5.13)

Although (5.12) is satisfiable in the simplest variable domain quantified modal logic with identity, its straightforward [7] **VK** *de re* counterpart

$$\neg(\exists x)(Holmes = x) \wedge (\exists x)B_a(Holmes = x) \tag{5.14}$$

is contradictory within a basic framework of necessary identity [8]. One option is to treat 'Holmes' as a non-rigid designator. Whilst technically this will mean that (5.14) is satisfiable, it provokes the question of why Holmes should be treated as a non-rigid designator in the first place. It also says that what Holmes refers to in the current world does not exist, which is not implausible, but also that what Holmes refers to in the doxastic alternatives does not exist in the current world, which is very much a dubious commitment. Furthermore, if we impose the negativity constraint, then the formula becomes unsatisfiable, for it dictates that what Holmes refers to in the current world must exist.

A circumvention of such problems and much better treatment of this issue is afforded by employment of the simplest of **IFVK** formulas. (5.13) is translated to

$$\neg(\exists x)(Holmes = x) \wedge B_a(\exists x_{/\{B\}})(Holmes = x) \tag{5.15}$$

[6] say, based on one of the **KD** systems, where the condition on frames of an epistemic logic expressed by $\mathrm{K}_a q \to q$ is replaced by $\mathrm{B}_a q \to q$. So each state accesses at least one doxastic alternative in my considerations.

[7] Hintikka translates $(\exists x)\mathrm{K}_a(b = x)$ as 'a knows who b is'. This may not be a definitive sense of 'knowing who', but it is a good place to start. See ([9], p. 167) for a discussion of this issue.)

[8] If a = b at world i, a = b at all other worlds.

which arguably is the best candidate thus far to capture the meaning of (5.13). The set of models which satisfy (5.15) are those which capture situations expressed by (5.13).

Extending this idea, consider the statement

$$\text{Everything is believed by } a \text{ to be } F \tag{5.16}$$

which translates to

$$(\forall x)\mathrm{B}_a F(x) \tag{5.17}$$

The assertion of this statement precludes the consistent assertion of the statement which translates to

$$(\exists x)\mathrm{B}_a \neg F(x) \tag{5.18}$$

Alternatively, it does not preclude the consistent assertion of the statement which translates to

$$\mathrm{B}_a(\exists x_{/\{B\}})\neg F(x) \tag{5.19}$$

These examples serve as justification for the employment of $\Box_1(\exists x/\{1\})$ type formulas over $(\exists x)\Box P(x)$ type formulas to express *de re* modality in certain situations. It seems harder to construct an example in which a slashed 'plausible' operator $P_{a/\{x\}}$ operator is used.

Nontheless, a useful example is apparent for the employment of $(\forall x)\Diamond_{1/\{x\}}P(x)$ over $\Diamond(\forall x)P(x)$. With constant domain quantified modal logic, the alethic expressions

$$\text{It is possible that everything is } F \tag{5.20}$$

and

$$\text{Everything is possibly } F \tag{5.21}$$

can be translated into $\Diamond(\forall x)P(x)$ and $(\forall x)\Diamond P(x)$ respectively. However, the tight correlation between (5.20) and $\Diamond(\forall x)P(x)$ gives way to ambiguity when it comes to a variable domain of quantification. With a variable domain of quantification, a means to logically capture the following qualified version of (5.20) is definitely worth having. If what is meant by (5.20) is something like

$$\text{It is possible that everything which actually exists is } F \tag{5.22}$$

then $\Diamond(\forall x)P(x)$ is insufficient. An adequate translation of (5.22) is $(\forall x)\Diamond_{/\{x\}}P(x)$. Surely the availability of such logical vocabulary to make such distinctions is advantageous.

Perhaps a most telling context for the consideration of informational independence is temporal discourse and logic. A domain increasing constraint validates the implausible claim that if something exists now, it will exist/has existed for all future/past times. A domain decreasing constraint validates the implausible claim that nothing will exist/has existed unless it exists now. Given this, one definitely apt province for variable domain quantification is quantified temporal logic.

Consider expressions involving temporal operators and quantification, where the quantification refers to the set of currently existing (or non-existing individuals), independent of the time indicated by the temporal operator. Contrast the two sentences:

$$\text{At some time in the future everything will be A} \tag{5.23}$$

$$\text{At some time in the future everything which } \textit{currently} \text{ exists will be A} \tag{5.24}$$

An adequate translation of 5.23 is $\langle F\rangle(\forall x)A(x)$. But an adequate translation of 5.24 is $(\forall x)\langle F\rangle_{/\{x\}}A(x)$. Similarly with the contrast between the two sentences:

$$\text{At some time in the past everything was A} \tag{5.25}$$

$$\text{At some time in the past everything which } \textit{currently} \text{ exists was A} \tag{5.26}$$

An adequate translation of 5.25 is $\langle P\rangle(\forall x)A(x)$. But an adequate translation of 5.26 is $(\forall x)\langle P\rangle_{/\{x\}}A(x)$.

Furthermore, since we are free to quantify over predicates which are not within the scope of the modal operator, we can attribute a property specifically to all currently existing objects, allowing us to construe the logical form of the sentence

$$\text{At some time in the future, every currently existing A will be B} \tag{5.27}$$

to be the **IFVK** formual $(\forall x)(A(x) \rightarrow \langle F\rangle_{/\{x\}}B(x))$

This theme can be extended to expressions involving universal temporal operators and existential individual quantification. The logical form of the statement

$$\text{There is something which will always be A.} \tag{5.28}$$

is $(\exists x)[F]A(x)$, whilst the statement

$$\text{It is always going to be the case that something is } A \tag{5.29}$$

could mean either $[F](\exists x)A(x)$ or $[F](\exists x_{/\{[F]\}})A(x)$

An adequate translation of 5.23 is $(\exists x)[F]A(x)$. But an adequate translation of 5.24 is $[F](\exists x_{/\{[F]\}})A(x)$.

The availability of **IFVK** terminology might also be used to good effect by a temporal reasoning system to resolve inconsistency. For example, if the datum $(\forall x)\langle F\rangle\neg B(x)$ is present in a system to which is added the datum $(\exists x)[F]B(x)$, an inconsistent state of affairs results. A resolution of this inconsistency by changing one of the formulas might result in the state of affairs which was actually intended by the input. $(\exists x)[F]B(x)$ could be swapped for $[F](\exists x_{/\{[F]\}}B(x))$.

These are just some cursory examples which await further development.

5.4

Hintikka is keen on advocating the idea that independence offers a new tool for discussing epistemic logic and the logic of questions and answers. In ([3], p. 83. and [4], p. 377.), he provides some cursory examples of formalising epistemic statements. Since he does not formally specify a background logic, it is difficult to definitively judge the correctness of what he says. I am very sure that the epistemic predicate logic he outlines in his seminal work *Knowledge and Belief* is effectively a variable domain quantified modal logic, yet in his discussion he acknowledges the equivalence between $(\exists x)\mathrm{K}S(x)$ and $\mathrm{K}(\exists x_{/\mathrm{K}})S(x)$, from which one could make the assumption that his discussion is within a constant domain context.

He considers a construction where the choice of the operative variable depends on the choice of a universal quantifier, as in:

$$\text{It is known whom each person admires most} \tag{5.30}$$

and claims that an adequate representation of this knowledge statement requires informational independence. A suitable translation of (5.30) is:

$$\mathrm{K}(\forall x)(\exists y/_{\mathrm{K}})A(x, y) \tag{5.31}$$

which he claims does not allow easily for a formulation in terms of a linear sequence of quantifiers plus K.

> If we try to express it on the first-order level without independent quantifiers, we run into an unsolvable dilemma. Since $(\exists y)$ depends on $(\forall x)$, it should come later than $(\forall x)$. But since it is independent of K, it should precede K and hence also $(\forall x)$.([3], pg. 85.)

Given a constant domain of quantification, the claim that "we run into an unsolvable dilemma" is incorrect, at least from the perspective of being able to find a truth-preserving formulation in terms of a linear sequence of quantifiers plus K. We saw in Theorem 3.2 that we can swap the $\forall$ and K, so 5.31 can be translated to

$$(\forall x)(\exists y)\mathrm{K}A(x,y). \tag{5.32}$$

In doing so however, the $\forall$ and K must be syntactically swapped. With the logical form of the original formula, the $(\forall x)$ has *wide scope* relative to K. With the logical form of the translation, the $(\forall x)$ has *narrow scope* relative to K. So despite this dispensability of the slash notation, its elimination may take place at the expense of the syntactic structure of the formula and its 'fit' with the form of the natural language statement it represents. Hence (5.30) may be better represented by (5.31) rather than (5.32).

One reason why this might be significant can be gathered from Hintikka's suggestion that the logical counterpart of the question ingredient of an indirect question in natural language can be a formula exhibiting informational independence. "Any knowledge statement can serve as the *desideratum* of a direct question, i.e. as a description of the cognitive state the questioner wants to achieve by his or her question. For example, (5.31) is the desideratum of a question of the form

Whom does each person admire most?

If the slash expression is removed from the questioned ingredients, the *presupposition* of the question is obtained. For example, the presupposition of (5.31) is

$$\mathrm{K}(\forall x)(\exists y)A(x,y) \tag{5.33}$$

Similarly, although $\Box(\forall x)(\exists y_{/\{\Box\}})A(x,y)$ and $(\forall x)(\exists y)\Box A(x,y)$ are equivalent in a constant domain of quantification, the former is seen as having *de dicto* necessity modality relative to the $(\forall x)$ and the latter is seen as having *de re* necessity modality relative to the $(\forall x)$.

Of course, this difference becomes more significant in the context of variable domain quantification, where the difference is both syntactical and model-theoretical. So if an epistemic first-order logic has a variable domain of quantification, then (5.31) arguably serves as a better representation than (5.32) for the logical form of (5.30). There are bound to be other good examples to be thought of and added to those discussed in Section 5.3. One that comes to mind is this; to say of every future thing that there is one thing in particular which they will always stand in a certain relation to, we realise that what is meant is not $[F](\forall x)(\exists y)M(x,y)$ but rather $[F](\forall x)(\exists y_{/\{[F]\}})M(x,y)$

5.5

Some philosophers, notably Quine, have argued that *de re* constructions, in which a quantifier outside an intensional context binds a variable occurring within an intensional context are problematic. We cannot, he claims, straightforwardly *quantify into* a belief or other intensional context. Or to put it in slightly different terms, intensional operators appear to block the interior reach of exterior quantifiers. He has provided arguments against both *de re* necessity and *de re* propositional attitude descriptions. Given the recent planetary reclassification of Pluto, consideration of Quine's argument concerning the number of planets in the solar system is apt. Consider:

1. $8 =$ number of planets
2. $\Box 8 > 7$
3. $\Box$ the number of planets is greater than 7.

(3) does not follow from (1) and (2). So the necessity operator induces *referential opacity.* Consider the statement.

$$(\exists x)\Box(x > 7) \tag{5.34}$$

For which object x is (5.34) rendered true? 8 or the 'number of planets'? Well since 8 is just the 'number of planets', one should say that x is both 8 and the 'number of planets'. But specify the object as '8' and the condition is rendered true; specify the object as 'the number of planets' and the condition is rendered false. We here have a failure of substitution. Whether or not the claim is necessay depends, not on the thing talked about, but on the way in which it is specified. If so Quine argues, there can be no clear understanding of whether an open sentence like $x > 7$ is necessarily true or not.

It is clear that validity of the following argument is undesirable:

$$(\Box(8 > 7) \wedge 8 = \#planets) \vdash \Box(\#planets > 7) \tag{5.35}$$

The simplest identity in quantified modal logic of rigid designation and identity invariance validates this argument. This problem is addressed by incorporating non-rigid designators into the language, which have a world-variant denotation. In this case, #planets is a non-rigid designator, that may change its denotation from world to world. This technique is well established in the literature.[9] With this technique, treating #planets as a non-rigid designator results in (5.35) not being valid. Although, whether or not this technical device eradicates philosophical concerns about quantifying into modal contexts and the failure of objectual satisfaction I am not in a position to authoritatively comment.

I do note however that independence friendly quantified modal logic may be of some use, since a simple informationally independence formula affords a way to express the model-definability restriction imposed by a *de re* formula without the need to quantify in. So the choice for an object x does not need to be made in an intensional context. The formula relevant to this issue would be

$$\Box(\exists x_{/\{\Box\}})(x > 7) \tag{5.36}$$

For (5.36) to be true, at each world accessed by the actual world, it must be the case that $(\exists x_{/\{\Box\}})(x > 7)$ is true. So there must be an x in each accessed world such that $x > 7$ is true. Furthermore, the chosen x must be the same for each world. Consider the following model:

- $W = \{w_0, w_1, w_2\}$
- $w_0Rw_0, w_0Rw_1, w_0Rw_2$
- $D = \{\delta_3, \delta_8, \delta_5\}$
- $v_{w_0}(3) = \delta_3, v_{w_0}(5) = \delta_5, v_{w_0}(8) = \delta_8, v_{w_0}(\#planets) = \delta_8$
- $v_{w_1}(3) = \delta_3, v_{w_1}(5) = \delta_5, v_{w_1}(8) = \delta_8, v_{w_1}(\#planets) = \delta_3$
- $v_{w_2}(3) = \delta_3, v_{w_2}(5) = \delta_5, v_{w_2}(8) = \delta_8, v_{w_2}(\#planets) = \delta_5$

The difference between the semantical games corresponding to (5.34) and (5.36) might help clarify matters.

In the game corresponding to (5.34), one of the names 8 or #planets can be firstly chosen. If 8 is chosen for x, then for all worlds which Falsifier can choose, $x > 7$ is true. If #planets is chosen for x, then if Falsifier chooses w_0, $x > 7$ is true and for w_1 and w_2, $x > 7$ is false. With (5.36), the choice by Falsifier of a world is made first. Secondly, the choice for x by Verifier is made in an extensional context and immediately either does or does not satisfy $x > 7$. The choice of an object which satisfies $x > 7$ is made relative to each world, so the truth of $x > 7$ does not depend on the choice of a binding modal operator, yet must also be the same for each world.

5.6

Most of the discussion in this section has focused on informational independence within a variable domain context. However, given the observations made earlier concerning the greater expressive power of **IFCK** relative to **CK**, an example within a constant domain context is in order. It is conceivable that there

[9] See ? and ?

are situations which can be better modeled with an **IFCK** formula rather than a **CK** formula. Here is a mundane example, which should nontheless suffice for the sake of example.

Consider a set of situations involving restaurants, people and meals. The domain of quantification, which to keep things simple is unrealistically small, contains two restaurants, two people and two dishes. Furthermore, each of the two restaurants has each of the two dishes on their menu. We want to analyse certain statements concerning the knowledge of a restaurant critic named Gil about who likes what and where, using the predicate $L(x, y, z)$ to express 'y likes z at x'. The logical form of the statement:

Gil knows that there is some restaurant such that everybody likes some menu dish at that restaurant (5.37)

is the **CK** formula

$$\mathrm{K}_G(\exists x)(\forall y)(\exists z)L(x, y, z) \tag{5.38}$$

Consider the following two models, which model situations captured by (5.37). Both of these models also rightly satisfy (5.38).

- $\mathcal{M}_1$
  - $W = \{w_0, w_1, w_2\}$
  - $w_0 R w_1, w_0 R w_2$
  - $D = \{Restuarant1, Restuarant2, Person1, Person2, Meal1, Meal2\}$
  -

x	y	z	w	$v_w(L(x, y, z))$
$restaurant1$	$Person1$	$Meal1$	w_1	1
$restaurant1$	$Person2$	$Meal2$	w_1	1
$restaurant2$	$Person1$	$Meal2$	w_2	1
$restaurant2$	$Person2$	$Meal1$	w_2	1

  * For every x, y, z, w permutation not tabulated above, $v_w(L(x, y, z)) = 0$

- $\mathcal{M}_2$
  - $W = \{w_0, w_1, w_2\}$
  - $w_0 R w_1, w_0 R w_2$
  - $D = \{Restuarant1, Restuarant2, Person1, Person2, Meal1, Meal2\}$
  -

x	y	z	w	$v_w(L(x, y, z))$
$restaurant1$	$Person1$	$Meal1$	w_1	1
$restaurant1$	$Person2$	$Meal2$	w_1	1
$restaurant2$	$Person1$	$Meal1$	w_2	1
$restaurant2$	$Person2$	$Meal2$	w_2	1

  * For every x, y, z, w permutation not tabulated above, $v_w(L(x, y, z)) = 0$

Now, say that as well as possessing the knowledge expressed by (5.37), Gil also knows which meal in particular each person likes, but may not know which restaurant in particular it is at which they like the meal. So, as well as 5.38

Gil knows *which* meal everybody likes (5.39)

The first model does not capture this situation while the second model does capture this situation (With the second model, Gil knows that Person1 likes Meal1 and Person2 likes Meal2). Therefore, a formula capturing the logical form of this set of situations must distinguish between $\mathcal{M}_1$ and $\mathcal{M}_2$ and in general, between both types of situations. In order to achieve this, we must employ **IFCK** terminology. The **IFCK** formula

$$\mathrm{K}_G(\exists x)(\forall y)(\exists z_{/\{\mathrm{K}\}})L(x, y, z) \tag{5.40}$$

is not satisfied by $\mathcal{M}_1$ but is satisfied by $\mathcal{M}_2$ and in general will be satisfied only by models which capture the second type of situation, so it does a better job of capturing 5.38 + 5.39.

This example shows that there is some utility in partially ordered quantification between quantifiers and modal operators even in the case of constant domain quantified modal logic. In fact, the need for independence indication to model such situations with **IFCK** is in a sense greater than the need for independence indication in **IF-FOL**. With **IF-FOL**, over finite domains of quantification, quantifiers can technically be removed in much the same way they can be removed with **FOL**; by reducing quantifiers to a collection of propositions consisting of the instantiations of predicates which are joined by logical connectives in such a way as to appropriately define the model. This strategy however is not available in the case of **IFCK** because the only means of quantifying over worlds are the modal operators.

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